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Nonlinear inertial amplifier liquid column dampers

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ABSTRACT

Conventional Tuned Liquid Column Dampers effectively mitigate structural vibrations; however, their efficacy is limited by the need for supplementary static mass, which escalates costs and complicates adaptability. This work presents a novel solution: Nonlinear Inertial Amplifier Liquid Column Dampers, which improve vibration attenuation without augmenting static mass. The suggested system is implemented on a single-degree-of-freedom framework, with its governing equations determined by Newton's second law and Lagrange's approach. The optimal design parameters are derived using H_2 and H_∞ optimisation methods. Results indicate that nonlinear inertial amplifier liquid column dampers much exceed the performance of conventional tuned liquid column dampers and inerter-based tuned liquid column dampers, attaining a dynamic response reduction of up to 83.60% and 82.11%. The closed-form solutions and parametric analyses validate the efficacy of this method, establishing nonlinear inertial amplifier liquid column dampers as a potential technique for enhancing structural resilience in civil engineering applications.

1. Introduction

A passive vibration control device called a Tuned Liquid Column Damper lessens structural vibrations brought on by outside factors like wind or seismic activity [1]. It consists of a vertical tube that is normally installed within a building or other structure enclosing a partially filled liquid column. The basic idea behind a Tuned Liquid Column Damper is to counteract the resonance frequencies of the structure by using the intrinsic frequency of the liquid column [2]. The liquid inside the Tuned Liquid Column Damper oscillates in response to external pressures that cause the structure to shake, but it does so with a phase difference that effectively dampens the vibrations of the structure. Engineers can adjust the characteristics of the liquid column (such as liquid volume, column height, and liquid density) to customise the Tuned Liquid Column Damper to target specific frequencies of structural vibration [3]. The main advantages of Tuned Liquid Column Dampers are their effectiveness in reducing structural vibrations, simplicity, and dependability. Compared to active vibration control systems, they are simpler to install, maintain, and operate passively and need no external power source. Tall buildings, bridges, and industrial facilities are just a few of the structures that have benefited from applying Tuned Liquid Column Dampers to increase their dynamic load resistance [4]. They are especially helpful in buildings with sensitive machinery or occupants since high vibrations can hurt people or cause structural damage or functional impairment [5].

For this damper to operate with durability, proper design is required. For optimal design purposes, a number of analytical, numerical, and simulation-based methods are available. Among them, the analytical techniques are the most reliable and effective. The H_2 optimisation method is commonly used in control system development, especially for passive vibration isolation devices, and

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is most likely related to control engineering. One type of passive vibration isolation device is the Tuned Liquid Column Damper. This means that this approach is appropriate. Precise closed-form formulas for the ideal design parameters of an isolator's natural frequency and damping ratio under random-white noise excitations are produced using the H_2 optimisation approach [6]. The greatest standard deviation of structural dynamic responses is decreased by H_2 optimised Tuned Liquid Column Damper. It is crucial to remember that there are various approaches to building control systems for passive vibration isolation systems; the H_2 optimisation method is just one of them [7]. Additional often employed methods encompass H_∞ control, Linear-Quadratic Regulator, and Model Predictive Control, which could offer differing compromises concerning efficacy, resilience, and intricacy of implementation. Numerous elements, such as the particular requirements of the application and the available resources, influence the chosen strategy. The most well-known and established analytical optimisation method is H_∞ optimisation [8]. Unlike the H_2 optimisation approach, which lowers the standard deviation of the dynamic responses, H_∞ optimised dampers lower the maximum amplitudes of the dynamic responses of the controlled structures. The optimisation strategy can be used with controlled structures that are harmonically stimulated. This idea was first proposed by Den Hartog in 1985 when his book "Mechanical Vibration" was released [9]. Some research examines the interrelated transverse and torsional vibrations of a mechanical system consisting of two similar beams and one firmly linked beam, emphasising their dynamic characteristics [10]. It illustrates how symmetries in these systems may facilitate the computation of eigenvalues and eigenmodes, resulting in less computing effort and design duration [11]. The study elucidates the characteristics of these coupled systems, presenting practical applications for enhancing the efficacy of vibration analysis in engineering and industrial design [12]. Researchers have developed a way to optimise the computation of eigenmodes in mechanical systems including bars, aiming to enhance the comprehension of deformations and stresses within the system's components via a semi-analytical model [13]. The nonlinear behaviour of systems necessitates meticulous numerical analysis, and the finite element method is especially adept at tackling the complexities related to nonlinear dynamics and thermal interactions [14,15].

Tuned Liquid Column Damper and tuned mass dampers' dynamic response reduction capabilities have historically been reinforced by raising the static mass of the dampers. However, as the dampers' natural frequency drops, increasing their static mass increases their flexibility. The damper's operational time increases as a result. The time duration of the dampers may greatly extend for large amplitude vibrations, potentially resulting in structural damage to the dampers [16]. In particular, the controlled structure's capacity to support loads declines. The flexibility of the dampers decreases from near-fault to far-field ground motion patterns. On the other hand, to reduce the dynamic reactions of the structures, increasing the static mass of the dampers necessitates increasing the number of tuned liquid column dampers. Consequently, the damper array's total static mass rises, increasing its cost. Consequently, it is necessary to increase the effective mass of the dampers instead of their static mass [17].

In order to reduce the dynamic reactivity of dynamic systems, Smith has invented inerters, an efficient mass amplification device that applies the force-to-current analogy [18]. In order to increase the energy dissipation capacity of conventional passive vibration control devices, inerters are installed within or in parallel with them. They work by multiplying the huge effective mass by rotating mass with motion transformers within the system. The adjusted liquid column dampers have already had the inerters added to them to boost their ability to reduce vibration [19]. Nevertheless, the inerter's uses do not lessen the shortcomings of the conventional Tuned Liquid Column Damper. To enhance the dynamic response reduction capabilities and lessen its disadvantages, the researchers have thus looked at mechanical devices, negative stiffness, alternative mass amplification, and negative mass for the Tuned Liquid Column Dampers. A kind of mass amplification device called an inertial amplifier may have a broadband gap at low frequencies, which qualifies it for application in civil engineering projects [20]. Moreover, inertial amplifiers are employed to lessen the dynamic reactivity of civil engineering structures. Passive dampers that are designed to lessen the reactivity of dynamic systems, such as wind turbines, bridges, and buildings, are incorporated with inertial amplifiers. Inertial amplifiers have never been used to address the other shortcomings of tuned liquid column dampers. There are no state-of-the-art designs that have the Tuned Liquid Column Damper's optimal design parameters in terms of the exact closed-form expressions. Consequently, the scope of the research has been determined.

To address the above-mentioned issues, this study introduces new nonlinear inertial amplifier liquid column dampers. The basic idea is to add nonlinear inertial amplifiers to conventional tuned liquid column dampers without adding static mass in order to boost their ability to reduce vibration. The application of the nonlinear inertial amplifier to the conventional Tuned Liquid Column damper mitigates the overflow of the liquid inside the container during vibration. This research distinguishes itself by including Nonlinear Inertial Amplifiers into Tuned Liquid Column Dampers (TLCDs), in contrast to the previous efforts that concentrate on vibration control enhancement via Tuned Mass Dampers (TMDs) [21]. The methods and ideas are tailored to accommodate the unique dynamics of TLCDs, which function differently from mass dampers. TLCDs use liquid motion to mitigate vibrations, and their functionality is intrinsically more complex owing to fluid dynamics and possible nonlinearities. TMDs use a mass-spring mechanism, resulting in unique behaviour and optimisation. This study is innovative in several significant aspects:

- Integration of Nonlinear Inertial Amplifiers into Tuned Liquid Column Dampers (TLCDs): This study extends the notion of inertial amplifiers to liquid-based dampers, including complexities related to fluid-structure interaction and the nonlinearities resulting from liquid dynamics.
- Enhanced Nonlinear Damping for Tuned Liquid Column Dampers (TLCDs): The research concentrates on enhancing both H_2 and H_∞ methodologies for nonlinear dampers inside a liquid-based system, representing an innovative use of techniques formerly utilised for mass dampers.
- Expansion to Viscous Damping Systems: This work presents a novel array of optimisation methodologies tailored for liquid column-based systems, enhancing dynamic response minimisation without augmenting static mass. The use of nonlinear inertial amplification is especially appropriate for civil engineering applications where mass limitations are critical.

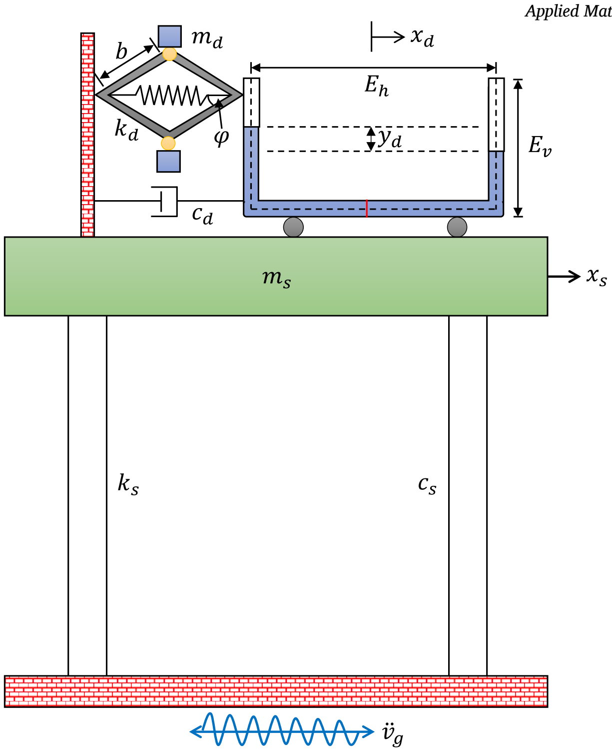

Fig. 1. A structure is equipped with an inertially amplified liquid column damper subjected to base excitation.

A single-degree-of-freedom system is equipped with this unique liquid damper to reduce its dynamic responses as it vibrates. The governing equations of motion for the controlled single-degree-of-freedom system are developed using Newton's second law and Lagrange's method. Then, the optimal closed-form design parameters are found using H_2 and H_∞ optimisation procedures. A parametric study is conducted using these optimal closed-form solutions. In order to compute the dynamic responses of the controlled structures analytically, the frequency response functions are constructed. The harmonic excitation is used at the base of the controlled structures as a loading function. The maximal dynamic responses of structures controlled by new dampers are compared with those controlled by conventional dampers in order to evaluate performance enhancement.

2. Structural model

A nonlinear inertial amplifier liquid column damper is installed at the top of a structure, and the damper-structure coupled figure is shown in Fig. 1. The structure is conceptualised as a single degree of freedom system (SDOF) having a mass of m_s, stiffness k_s, and damping c_s. The damper is oscillated in two directions due to the application of high-amplitude vibration. As a result, the connections of the amplifier are deflected, and the effective mass of the nonlinear inertial amplifier liquid column damper [22] has been derived as

$$m_{cd} = \underbrace{m_c + 0.5 m_d \left(1 + \frac{1}{\tan^2 \varphi}\right)}_{m_{cd0}} + \underbrace{\frac{\cos \varphi m_d}{2b \sin^4 \varphi}}_{m_{cd1}} v_d + \underbrace{\frac{\left(1 + 3\cos^2 \varphi\right) m_d}{4b^2 \sin^6 \varphi}}_{m_{cd2}} v_d^2, \tag{1}$$

where m_c defines the static mass of the damper, i.e. $m_c = m_a + m_w$. m_a defines the static mass of the container. m_w defines the static mass of the liquid inside the container, which is derived as $m_w = \rho A E$, where ρ defines the density of the liquid. A defines the cross-sectional area of the damper. E defines the total length of the liquid container. m_d defines the mass of the inertial amplifier. φ defines the inertial angle. b defines the length of the amplifier link. If the low amplitude vibration is applied at the base instead of the high amplitude vibration, the deflection in b will become zero. Simultaneously, m_{cd1} and m_{cd2} become zero, i.e. $m_{cd1} = 0$ and $m_{cd2} = 0$. x_s defines the absolute deflection of the SDOF system. x_d defines the lateral deflection of the container with liquid mass. y_d defines the vertical deflection of the liquid mass. $\ddot{v}_g$ defines base excitation.

3. Equations of motion and derivations of optimal design parameters

The equation of motion of the SDOF system equipped with a nonlinear inertial amplifier liquid column damper is derived using Newton's second law and expressed as

$$
\begin{aligned}
&m_s\ddot{x}_s = -k_s\left(x_s - v_g\right) - c_s\left(\dot{x}_s - \dot{v}_g\right) + k_{cd}\left(x_d - x_s\right) + c_{cd}\left(\dot{x}_d - \dot{x}_s\right),\\
&m_s\ddot{v}_s + c_s\dot{v}_s + k_s v_s - c_{cd}\dot{v}_d - k_{cd}v_d = -m_s\ddot{v}_g,
\end{aligned} \tag{2}
$$

where $v_s = x_s - v_g$ and $v_d = x_d - x_s$ define the relative deflection of the SDOF system and damper. c_{cd} and k_{cd} define the effective damping and stiffness of the damper when the mass amplification effect of the nonlinear inertial amplifier is applied to the damper. The effective damping and stiffness have been derived as

$$
c_{cd} = 2m_{cd}\xi_d\epsilon_d \quad \text{and} \quad k_{cd} = m_{cd}\epsilon_d^2. \tag{3}
$$

In a similar way, the equations of motion of the damper by considering its lateral and vertical motions during vibration have been derived as

$$
\begin{aligned}
&m_{cd}\ddot{v}_s + m_{cd}\ddot{v}_d + m_{dh}\ddot{w}_d + c_{cd}\dot{v}_d + k_{cd}v_d = -m_{cd}\ddot{v}_g,\\
&m_{dh}\ddot{v}_s + m_{dh}\ddot{v}_d + m_{de}\ddot{w}_d + \frac{m_{de}}{2E}\xi_w|\dot{w}_d|\dot{w}_d + \frac{2m_{de}g}{E}w_d = -m_{dh}\ddot{v}_g,
\end{aligned} \tag{4}
$$

where w_d defines the relative motion of the liquid inside the liquid column. m_{dh} and m_{de} define the effective masses of the liquid inside the horizontal portion and the total length of the nonlinear inertial amplifier liquid column damper.

$$
\begin{aligned}
&m_{dh} = \underbrace{\rho A E_h + 0.5m_d\left(1 + \frac{1}{\tan^2\varphi}\right)}_{m_{dh0}} + \underbrace{\frac{\cos\varphi m_d}{2b\sin^4\varphi}}_{m_{dh1}} w_d + \underbrace{\frac{\left(1+3\cos^2\varphi\right)m_d}{4b^2\sin^6\varphi}}_{m_{dh2}} w_d^2,\\
&m_{de} = \underbrace{m_w + 0.5m_d\left(1 + \frac{1}{\tan^2\varphi}\right)}_{m_{de0}} + \underbrace{\frac{\cos\varphi m_d}{2b\sin^4\varphi}}_{m_{de1}} w_d + \underbrace{\frac{\left(1+3\cos^2\varphi\right)m_d}{4b^2\sin^6\varphi}}_{m_{de2}} w_d^2.
\end{aligned} \tag{5}
$$

Equation (1) and Eq. (5) are substituted in Eq. (2) and Eq. (5).

$$
\begin{aligned}
&m_s\ddot{v}_s + c_s\dot{v}_s + k_s v_s - 2m_{cd0}\xi_d\epsilon_d\dot{v}_d - 2m_{cd1}\xi_d\epsilon_d v_d\dot{v}_d - 2m_{cd2}\xi_d\epsilon_d v_d^2\dot{v}_d - m_{cd0}\epsilon_d^2 v_d\\
&- m_{cd1}\epsilon_d^2 v_d^2 - m_{cd2}\epsilon_d^2 v_d^3 = -m_s\ddot{v}_g,\\
&m_{cd0}\ddot{v}_s + m_{cd1}v_d\ddot{v}_s + m_{cd2}v_d^2\ddot{v}_s + m_{cd0}\ddot{v}_d + m_{cd1}v_d\ddot{v}_d + m_{cd2}v_d^2\ddot{v}_d + m_{dh0}\ddot{w}_d\\
&+ m_{dh1}w_d\ddot{w}_d + m_{dh2}w_d^2\ddot{w}_d + 2m_{cd0}\xi_d\epsilon_d\dot{v}_d + 2m_{cd1}\xi_d\epsilon_d v_d\dot{v}_d + 2m_{cd2}\xi_d\epsilon_d v_d^2\dot{v}_d\\
&+ m_{cd0}\epsilon_d^2 v_d + m_{cd1}\epsilon_d^2 v_d^2 + m_{cd2}\epsilon_d^2 v_d^3 = -m_{cd0}\ddot{v}_g - m_{cd1}v_d\ddot{v}_g - m_{cd2}v_d^2\ddot{v}_g,\\
&m_{dh0}\ddot{v}_s + m_{dh1}w_d\ddot{v}_s + m_{dh2}w_d^2\ddot{v}_s + m_{dh0}\ddot{v}_d + m_{dh1}w_d\ddot{v}_d + m_{dh2}w_d^2\ddot{v}_d + m_{de0}\ddot{w}_d\\
&+ m_{de1}w_d\ddot{w}_d + m_{de2}w_d^2\ddot{w}_d + \frac{m_{de0}}{2E}\xi_w|\dot{w}_d|\dot{w}_d + \frac{m_{de1}}{2E}w_d\xi_w|\dot{w}_d|\dot{w}_d + \frac{m_{de2}}{2E}w_d^2\xi_w|\dot{w}_d|\dot{w}_d\\
&+ \frac{2m_{de0}g}{E}w_d + \frac{2m_{de1}g}{E}w_d^2 + \frac{2m_{de2}g}{E}w_d^3 = -m_{dh0}\ddot{v}_g - m_{dh1}w_d\ddot{v}_g - m_{dh2}w_d^2\ddot{v}_g.
\end{aligned} \tag{6}
$$

Equation (6) is a highly nonlinear equation. The statistical linearisation approach is used to linearise each nonlinear component of equation Eq. (6) in order to apply the H_2 and H_∞ optimisation techniques. The suggested analytical technique is a precise optimisation process that does not include approximations. In addition, the statistical linearisation technique has been exclusively used to reduce the complex nonlinear equations of motion and to directly employ analytical optimisation approaches. The suggested analytical optimisation techniques may be used to derive the optimum design parameters using mathematical closed-form expressions. These expressions are more practical compared to the optimal design parameters acquired by the numerical approaches. The use of numerical optimisation techniques will not provide any further advantages to this study. The validity of the optimised design parameters for H_2 and H_∞ has been further confirmed by the study of the frequency and time domain response evaluations. It is important to acknowledge that the statistical linearisation approach is well recognised, and its precision is extensively supported by research. In order to do this, it has been determined that the controlled single degree of freedom system is exposed to zero mean Gaussian white noise excitation. Suppose an error is considered while applying the statistical linearisation method, and it can happen for the damping element of the liquid inside the container of the damper. The damping of the liquid is developed due to the presence of the relative velocity of the liquid inside the container during vibration, and the mathematically developed equations of the liquid damping are highly nonlinear. An error may occur during the transformation of the linearised term from the nonlinear damping element using the statistical linearisation method. The error can be estimated as

$$
\lambda = \frac{m_{de0}}{2E}\xi_w|\dot{w}_d|\dot{w}_d - \underbrace{\frac{2m_{de0}}{E}\xi_e}_{c_e}\dot{w}_d, \tag{7}
$$

where ξ_e defines the equivalent damping ratio of the liquid and c_e defines the equivalent damping of the liquid.

$$\frac{\partial \lambda^2}{\partial c_e} = E_j\left\{ \left(\frac{m_{de0}}{2E}\xi_w |\dot{w}_d| \dot{w}_d - c_e \dot{w}_d \right)^2 \right\} = 0 \quad \text{and} \quad \xi_e = \frac{\sigma_{\dot{w}_d}\xi_w}{\sqrt{2\pi}}. \tag{8}$$

Therefore, zero errors have been derived during the statistical linearisation process. Employing this method, other nonlinear elements of Eq. (6) have been linearised and expressed as

$$\begin{aligned}
c_{e1} &= E_j\left\{ \frac{\partial\left(2m_{cd1}\xi_d\epsilon_d v_d \dot{v}_d\right)}{\partial \dot{v}_d} \right\} = 0, c_{e2} = E_j\left\{ \frac{\partial\left(2m_{cd2}\xi_d\epsilon_d v_d^2 \dot{v}_d\right)}{\partial \dot{v}_d} \right\} = 0,\\
c_{de1} &= E_j\left\{ \frac{\partial\left(2m_{cd1}\xi_d\epsilon_d v_d \dot{v}_d\right)}{\partial \dot{v}_d} \right\} = 0, c_{de2} = E_j\left\{ \frac{\partial\left(2m_{cd2}\xi_d\epsilon_d v_d^2 \dot{v}_d\right)}{\partial \dot{v}_d} \right\} = 0,\\
c_{we1} &= E_j\left\{ \frac{\partial\left(\frac{m_{de1}}{2E} w_d \xi_w |\dot{w}_d| \dot{w}_d\right)}{\partial \dot{w}_d} \right\} = 0, c_{we2} = E_j\left\{ \frac{\frac{m_{de2}}{2E} w_d^2 \xi_w |\dot{w}_d| \dot{w}_d}{\partial \dot{w}_d} \right\} = 0.
\end{aligned} \tag{9}$$

$$\begin{aligned}
k_{e1} &= E_j\left\{ \frac{\partial\left(m_{cd1}\epsilon_d^2 v_d^2\right)}{\partial v_d} \right\} = 0, k_{e2} = E_j\left\{ \frac{\partial\left(m_{cd2}\epsilon_d^2 v_d^3\right)}{\partial v_d} \right\} = 3m_{cd2}\epsilon_d^2\sigma_{v_d}^2,\\
k_{we1} &= E_j\left\{ \frac{\partial\left(\frac{2m_{de1}g}{E} w_d^2\right)}{\partial w_d} \right\} = 0, k_{we2} = E_j\left\{ \frac{\partial\left(\frac{2m_{de2}g}{E} w_d^3\right)}{\partial w_d} \right\} = \frac{6m_{de2}g}{E}\sigma_{w_d}^2.
\end{aligned} \tag{10}$$

$$\begin{aligned}
m_{e1} &= E_j\left\{ \frac{\partial\left(m_{cd1} v_d \ddot{v}_s\right)}{\partial \ddot{v}_s} \right\} = 0, m_{e2} = E_j\left\{ \frac{\partial\left(m_{cd2} v_d^2 \ddot{v}_s\right)}{\partial \ddot{v}_s} \right\} = 0,\\
m_{de1} &= E_j\left\{ \frac{\partial\left(m_{cd1} v_d \ddot{v}_d\right)}{\partial \ddot{v}_d} \right\} = 0, m_{de2} = E_j\left\{ \frac{\partial\left(m_{cd2} v_d^2 \ddot{v}_d\right)}{\partial \ddot{v}_d} \right\} = 0,\\
m_{he1} &= E_j\left\{ \frac{\partial\left(m_{dh1} w_d \ddot{w}_d\right)}{\partial \ddot{w}_d} \right\} = 0, m_{he2} = E_j\left\{ \frac{\partial\left(m_{dh2} w_d^2 \ddot{w}_d\right)}{\partial \ddot{w}_d} \right\} = 0,\\
m_{ge1} &= E_j\left\{ \frac{\partial\left(m_{cd1} v_d \ddot{v}_g\right)}{\partial \ddot{v}_g} \right\} = 0, m_{ge2} = E_j\left\{ \frac{\partial\left(m_{cd2} v_d^2 \ddot{v}_g\right)}{\partial \ddot{v}_g} \right\} = 0,\\
m_{se1} &= E_j\left\{ \frac{\partial\left(m_{dh1} w_d \ddot{v}_s\right)}{\partial \ddot{v}_s} \right\} = 0, m_{se2} = E_j\left\{ \frac{\partial\left(m_{dh2} w_d^2 \ddot{v}_s\right)}{\partial \ddot{v}_s} \right\} = 0,\\
m_{we1} &= E_j\left\{ \frac{\partial\left(m_{dh1} w_d \ddot{v}_d\right)}{\partial \ddot{v}_d} \right\} = 0, m_{we2} = E_j\left\{ \frac{\partial\left(m_{dh2} w_d^2 \ddot{v}_d\right)}{\partial \ddot{v}_d} \right\} = 0,\\
m_{de1} &= E_j\left\{ \frac{\partial\left(m_{dh1} w_d \ddot{w}_d\right)}{\partial \ddot{w}_d} \right\} = 0, m_{de2} = E_j\left\{ \frac{\partial\left(m_{dh2} w_d^2 \ddot{w}_d\right)}{\partial \ddot{w}_d} \right\} = 0,\\
m_{wge1} &= E_j\left\{ \frac{\partial\left(m_{dh1} w_d \ddot{v}_g\right)}{\partial \ddot{v}_g} \right\} = 0, m_{wge2} = E_j\left\{ \frac{\partial\left(m_{dh2} w_d^2 \ddot{v}_g\right)}{\partial \ddot{v}_g} \right\} = 0,
\end{aligned} \tag{11}$$

where E_j defines the expectation operator. Initially, the values of the σ_{v_d}, σ_{w_d}, and $\sigma_{\dot{w}_d}$ are considered zero [23] to derive the frequency response function of the controlled single degree of freedom system. Therefore, after applying these initial conditions, the governing equations of motion in Eq. (6) have been derived as

$$\begin{aligned}
&m_s\ddot{v}_s + c_s\dot{v}_s + k_s v_s - 2m_{cd0}\xi_d\epsilon_d\dot{v}_d - m_{cd0}\epsilon_d^2 v_d = -m_s\ddot{v}_g,\\
&m_{cd0}\ddot{v}_s + m_{cd0}\ddot{v}_d + m_{dh0}\ddot{w}_d + 2m_{cd0}\xi_d\epsilon_d\dot{v}_d + m_{cd0}\epsilon_d^2 v_d = -m_{cd0}\ddot{v}_g,\\
&m_{dh0}\ddot{v}_s + m_{dh0}\ddot{v}_d + m_{de0}\ddot{w}_d + \frac{2m_{de0}}{E}\xi_e\dot{w}_d + \frac{2m_{de0}g}{E}w_d = -m_{dh0}\ddot{v}_g.
\end{aligned} \tag{12}$$

Base excitations are applied at the base of the controlled single-degree-of-freedom system. The steady-state deflections of the single degree of freedom system, damper's container, and liquid have been considered $v_s = V_s e^{i\omega t}$, $v_d = V_d e^{i\omega t}$, $w_d = W_d e^{i\omega t}$, and $\ddot{v}_g = V_g e^{i\omega t}$. The first and second expressions of the Eq. (12) are divided by the mass of the SDOF system, i.e. m_s, and the third expression is divided by m_{de0}, which leads to

$$\begin{aligned}
&\ddot{v}_s + 2\xi_s\epsilon_s\dot{v}_s + \epsilon_s^2 v_s - 2\mu_{cd0}\xi_d\epsilon_d\dot{v}_d - \mu_{cd0}\epsilon_d^2 v_d = -\ddot{v}_g,\\
&\mu_{cd0}\ddot{v}_s + \mu_{cd0}\ddot{v}_d + \mu_{dh0}\ddot{w}_d + 2\mu_{cd0}\xi_d\epsilon_d\dot{v}_d + \mu_{cd0}\epsilon_d^2 v_d = -\mu_{cd0}\ddot{v}_g,\\
&\left(\frac{\mu_{dh0}}{\mu_{de0}}\right)\ddot{v}_s + \left(\frac{\mu_{dh0}}{\mu_{de0}}\right)\ddot{v}_d + \ddot{w}_d + \left(\frac{2}{E}\right)\xi_e\dot{w}_d + \underbrace{\left(\frac{2g}{E}\right)}_{\epsilon_w^2} w_d = -\left(\frac{\mu_{dh0}}{\mu_{de0}}\right)\ddot{v}_g.
\end{aligned} \tag{13}$$

After considering the initial condition for the damping ratio of the liquid and the steady state solutions for the entire controlled single degree of freedom system, the transfer function has been derived and expressed as

$$\begin{bmatrix} 2\xi_s\epsilon_s q+q^2+{\epsilon_s}^2 & A_{12} & 0 \\ \mu_{cd0}q^2 & A_{22} & \mu_{dh0}q^2 \\ \frac{\mu_{dh0}q^2}{\mu_{de0}} & \frac{\mu_{dh0}q^2}{\mu_{de0}} & q^2+{\epsilon_w}^2 \end{bmatrix}\begin{Bmatrix} V_s \\ V_d \\ W_d \end{Bmatrix} = -\begin{bmatrix} 1 \\ \mu_{cd0} \\ \frac{\mu_{dh0}}{\mu_{de0}} \end{bmatrix} V_g, \tag{14}$$

$$A_{12} = -2\mu_{cd0}\xi_d\epsilon_d q - \mu_{cd0}{\epsilon_d}^2, \quad \text{and} \quad A_{22} = 2\mu_{cd0}\xi_d\epsilon_d q + \mu_{cd0}q^2 + \mu_{cd0}{\epsilon_d}^2.$$

The dynamic response of the single degree of freedom system is derived using Eq. (14) and expressed as

$$H_s = \frac{V_s}{V_g} = \frac{\left(\begin{array}{c} -2{\mu_{cd0}}^2\mu_{de0}q^3\xi_d\epsilon_d - 2{\mu_{cd0}}^2\mu_{de0}q\xi_d\epsilon_d{\epsilon_w}^2 + 2\mu_{cd0}{\mu_{dh0}}^2q^3\xi_d\epsilon_d \\ -{\mu_{cd0}}^2\mu_{de0}q^2{\epsilon_d}^2 - {\mu_{cd0}}^2\mu_{de0}{\epsilon_d}^2{\epsilon_w}^2 - 2\mu_{cd0}\mu_{de0}q^3\xi_d\epsilon_d \\ -2\mu_{cd0}\mu_{de0}q\xi_d\epsilon_d{\epsilon_w}^2 + \mu_{cd0}{\mu_{dh0}}^2q^2{\epsilon_d}^2 - \mu_{cd0}\mu_{de0}q^4 \\ -\mu_{cd0}\mu_{de0}q^2{\epsilon_d}^2 - \mu_{cd0}\mu_{de0}q^2{\epsilon_w}^2 - \mu_{cd0}\mu_{de0}{\epsilon_d}^2{\epsilon_w}^2 + {\mu_{dh0}}^2q^4 \end{array}\right)}{\delta_d}. \tag{15}$$

In the same way, the dynamic response of the lateral deflection of the container with liquid mass is derived and expressed as

$$H_d = \frac{V_d}{V_g} = \frac{\left(-\left(2q\xi_s+\epsilon_s\right)\epsilon_s\left(\mu_{cd0}\mu_{de0}q^2 + \mu_{cd0}\mu_{de0}{\epsilon_w}^2 - {\mu_{dh0}}^2q^2\right)\right)}{\delta_d}. \tag{16}$$

The dynamic response of the vertical deflection of the liquid mass is derived as

$$H_{wd} = \frac{W_d}{V_g} = \frac{\left(-\epsilon_d\left(2q\xi_s+\epsilon_s\right)\epsilon_s\mu_{cd0}\mu_{dh0}\left(2q\xi_d+\epsilon_d\right)\right)}{\delta_d}. \tag{17}$$

The denominator of Eq. (15), Eq. (16), and Eq. (17) are derived as

$$\delta_d = \left(\begin{array}{c} \left(\mu_{cd0}\mu_{de0} - {\mu_{dh0}}^2\right)q^6 \\ +\left(\begin{array}{c} 2\xi_d{\mu_{cd0}}^2\mu_{de0}\epsilon_d - 2\mu_{cd0}{\mu_{dh0}}^2\xi_d\epsilon_d + 2\mu_{cd0}\mu_{de0}\xi_d\epsilon_d \\ +2\mu_{cd0}\mu_{de0}\xi_s\epsilon_s - 2\xi_s{\mu_{dh0}}^2\epsilon_s \end{array}\right)q^5 \\ +\left(\begin{array}{c} 4\mu_{cd0}\mu_{de0}\xi_d\xi_s\epsilon_d\epsilon_s + {\mu_{cd0}}^2\mu_{de0}{\epsilon_d}^2 - \mu_{cd0}{\mu_{dh0}}^2{\epsilon_d}^2 \\ +\mu_{cd0}\mu_{de0}{\epsilon_d}^2 + \mu_{cd0}\mu_{de0}{\epsilon_s}^2 + \mu_{cd0}\mu_{de0}{\epsilon_w}^2 - {\mu_{dh0}}^2{\epsilon_s}^2 \end{array}\right)q^4 \\ +\left(\begin{array}{c} 2{\mu_{cd0}}^2\mu_{de0}\xi_d\epsilon_d{\epsilon_w}^2 + 2\mu_{cd0}\mu_{de0}\xi_d\epsilon_d{\epsilon_s}^2 + 2\mu_{cd0}\mu_{de0}\xi_d\epsilon_d{\epsilon_w}^2 \\ +2\mu_{cd0}\mu_{de0}\xi_s{\epsilon_d}^2\epsilon_s + 2\mu_{cd0}\mu_{de0}\xi_s\epsilon_s{\epsilon_w}^2 \end{array}\right)q^3 \\ +\left(\begin{array}{c} 4\mu_{cd0}\mu_{de0}\xi_d\xi_s\epsilon_d\epsilon_s{\epsilon_w}^2 + {\mu_{cd0}}^2\mu_{de0}{\epsilon_d}^2{\epsilon_w}^2 + \mu_{cd0}\mu_{de0}{\epsilon_d}^2{\epsilon_s}^2 \\ +\mu_{cd0}\mu_{de0}{\epsilon_d}^2{\epsilon_w}^2 + \mu_{cd0}\mu_{de0}{\epsilon_s}^2{\epsilon_w}^2 \end{array}\right)q^2 \\ +\left(2\mu_{cd0}\mu_{de0}\xi_d\epsilon_d{\epsilon_s}^2{\epsilon_w}^2 + 2\mu_{cd0}\mu_{de0}\xi_s{\epsilon_d}^2\epsilon_s{\epsilon_w}^2\right)q + \mu_{cd0}\mu_{de0}{\epsilon_d}^2{\epsilon_s}^2{\epsilon_w}^2 \end{array}\right). \tag{18}$$

Equation (18) is a sixth order polynomial equation and the stochastic process with random white noise excitation is applied to derive the standard deviation of the deflection of the single degree of freedom system [24]. Using this standard deviation, the optimal design parameters for the governing system parameters of the damper such as natural frequency of the container with liquid, i.e. ϵ_d, natural frequency of the liquid, i.e. ϵ_w, and the damping ratio of the container with liquid, i.e. ξ_d is obtained in respect of closed-form expression.

3.1. H_2 optimisation

H_2 optimisation method is applied to perform this optimisation procedure. In addition, the total root mean square value of the deflection is minimised using H_2 optimised dampers. As a result, the damping ratio of the single degree of freedom system in Eq. (15) and Eq. (18) is considered zero, i.e. for this evaluation process. Therefore, the standard deviation of the SDOF system is obtained

$$\sigma_{v_s}^2 = \frac{S_0\pi N_1}{2\mu_{de0}\epsilon_d\xi_d\mu_{cd0}{\epsilon_w}^2\left(\mu_{cd0}\mu_{de0}{\epsilon_s}^2 - \mu_{cd0}\mu_{de0}{\epsilon_w}^2 - {\mu_{dh0}}^2{\epsilon_s}^2\right)^2{\epsilon_s}^6}. \tag{19}$$

The closed-form expression for N_1 is listed in Appendix A. Equation (19) is divided by ξ_d, ϵ_d, and ϵ_w and the mathematical expression is obtained as follows.

$$\frac{\partial\sigma_{v_s}^2}{\partial\xi_d} = 0, \frac{\partial\sigma_{v_s}^2}{\partial\epsilon_d} = 0, \quad \text{and} \quad \frac{\partial\sigma_{v_s}^2}{\partial\epsilon_w} = 0. \tag{20}$$

Equation (19) is substituted in the first expression of Eq. (20). Hence, the closed-form expression for ξ_d has been derived as

$$\xi_d = \sqrt{\frac{N_2}{N_3}}. \tag{21}$$

Equation (21) is substituted in Eq. (19) and the standard deviation is modified which is expressed as

$$
\sigma_{v_s}^2 = \frac{2S_0\pi \left(\begin{array}{c}
\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\mu_{cd0}{}^6 \\
+4\,\epsilon_w{}^2\left(\left(\epsilon_d{}^2+1/4\,\epsilon_s{}^2\right)\left(\epsilon_s+\epsilon_w\right)\left(\epsilon_s-\epsilon_w\right)\mu_{de0}-1/2\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^2\right) \\
\left(\epsilon_s+\epsilon_w\right)\mu_{de0}{}^2\left(\epsilon_s-\epsilon_w\right)\epsilon_d{}^2\mu_{cd0}{}^5 \\
+6\left(\begin{array}{c}
\epsilon_d{}^2\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\mu_{de0}{}^2+1/6\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^2 \\
+1/6\,\mu_{dh0}{}^2\left(\epsilon_s+\epsilon_w\right)\epsilon_s{}^2\left(\epsilon_s-\epsilon_w\right) \\
\left(\left(-9\,\epsilon_d{}^2-2\,\epsilon_s{}^2\right)\epsilon_w{}^2+\epsilon_d{}^2\epsilon_s{}^2\right)\mu_{de0}
\end{array}\right) \\
\mu_{de0}\,\epsilon_d{}^2\mu_{cd0}{}^4 \\
+4\left(\begin{array}{c}
\epsilon_w{}^2\left(\epsilon_s+\epsilon_w\right)^2\left(\epsilon_d{}^2-3/4\,\epsilon_s{}^2\right)\left(\epsilon_s-\epsilon_w\right)^2\mu_{de0}{}^2 \\
-3\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)\left(\epsilon_s+\epsilon_w\right)\mu_{de0} \\
-1/2\,\mu_{dh0}{}^4\epsilon_s{}^4\left(\left(-3\,\epsilon_d{}^2-1/2\,\epsilon_s{}^2\right)\epsilon_w{}^2+\epsilon_d{}^2\epsilon_s{}^2\right)
\end{array}\right) \\
\mu_{de0}\,\epsilon_d{}^2\mu_{cd0}{}^3 \\
+\left(\begin{array}{c}
\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\left(\epsilon_d-\epsilon_s\right)^2\left(\epsilon_d+\epsilon_s\right)^2\mu_{de0}{}^3 \\
-4\,\mu_{dh0}{}^2\left(\begin{array}{c}\left(-5/4\,\epsilon_d{}^2+3/2\,\epsilon_s{}^2\right)\epsilon_w{}^2 \\ +\epsilon_d{}^2\epsilon_s{}^2-3/2\,\epsilon_s{}^4\end{array}\right) \\
\epsilon_w{}^2\epsilon_s{}^2\epsilon_d{}^2\mu_{de0}{}^2 \\
+6\,\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2+\mu_{dh0}{}^6\epsilon_d{}^4\epsilon_s{}^6
\end{array}\right)\mu_{cd0}{}^2 \\
+2\,\mu_{dh0}{}^2\epsilon_w{}^2\left(\begin{array}{c}\left(\left(-\epsilon_d{}^2+\epsilon_s{}^2\right)\epsilon_w{}^2+\epsilon_d{}^2\epsilon_s{}^2-\epsilon_s{}^4\right)\mu_{de0} \\ -3/2\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^2\end{array}\right) \\
\epsilon_s{}^4\mu_{de0}\,\mu_{cd0} \\
+\mu_{de0}\,\mu_{dh0}{}^4\epsilon_s{}^8\epsilon_w{}^2
\end{array}\right)}{N_4}. \tag{22}
$$

The closed-form expression for N_4 is listed in Appendix A. Equation (22) is contained ϵ_d and ϵ_w, which needs to be extracted for the optimisation. To perform that, Eq. (22) is substituted in the second expression of Eq. (20). Hence, the closed-form expression for ϵ_d is derived as

$$
\epsilon_d = \sqrt{\frac{\begin{array}{c}\left(\left(\epsilon_s-\epsilon_w\right)\left(\epsilon_s+\epsilon_w\right)\left(\mu_{cd0}-2\right)\left(\mu_{cd0}+1\right)^2\mu_{de0}-\mu_{dh0}{}^2\epsilon_s{}^2\left(\mu_{cd0}{}^2-3\right)\right)\epsilon_s{}^2 \\ \left(\mu_{cd0}\,\mu_{de0}\,\epsilon_w{}^2+\mu_{dh0}{}^2\epsilon_s{}^2-\mu_{cd0}\,\mu_{de0}\,\epsilon_s{}^2\right)\mu_{de0}\,\epsilon_w{}^2\end{array}}{2\,\mu_{cd0}\left(\begin{array}{c}
\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\left(\mu_{cd0}+1\right)^4\mu_{de0}{}^3 \\
-\left(\begin{array}{c}\epsilon_s{}^2\left(\mu_{cd0}+1\right)\left(\mu_{cd0}{}^2+4\,\mu_{cd0}+2\right)\epsilon_w{}^2 \\ -1/2\,\mu_{cd0}{}^2\epsilon_s{}^4-\left(\mu_{cd0}+5/2\right)\left(\mu_{cd0}+1\right)^2\epsilon_w{}^4\end{array}\right)\mu_{de0}{}^2 \\
2\,\mu_{dh0}{}^2\epsilon_s{}^2 \\
+\left(\left(\mu_{cd0}{}^2+6\,\mu_{cd0}+6\right)\epsilon_w{}^2-2\,\mu_{cd0}\,\epsilon_s{}^2\right)\mu_{dh0}{}^4\epsilon_s{}^4\mu_{de0}+\mu_{dh0}{}^6\epsilon_s{}^6
\end{array}\right)}}. \tag{23}
$$

Equation (23) is substituted in Eq. (22) and the modified version of Eq. (22) is derived as

$$
\sigma_{v_s}^2 = \frac{\dfrac{\begin{array}{c}S_0\mu_{de0}\,\epsilon_w{}^2\epsilon_s{}^4\pi\left(\mu_{cd0}\,\mu_{de0}\,\epsilon_w{}^2+\mu_{dh0}{}^2\epsilon_s{}^2-\mu_{cd0}\,\mu_{de0}\,\epsilon_s{}^2\right)^2 \\ \left(\begin{array}{c}
\mu_{cd0}\,\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\left(\mu_{cd0}-4\right)\left(\mu_{cd0}+1\right)^4\mu_{de0}{}^3 \\
-2\left(\begin{array}{c}2\,\mu_{cd0}{}^2\epsilon_s{}^4-\left(\mu_{cd0}-4\right)\left(\mu_{cd0}+1\right)^4\epsilon_w{}^4 \\ +\epsilon_s{}^2\left(\mu_{cd0}+1\right)\left(\mu_{cd0}{}^4-\mu_{cd0}{}^3-9\,\mu_{cd0}{}^2-13\,\mu_{cd0}-2\right)\epsilon_w{}^2\end{array}\right) \\
\epsilon_s{}^2\mu_{dh0}{}^2\mu_{de0}{}^2 \\
+\epsilon_s{}^4\left(\left(\mu_{cd0}{}^4-10\,\mu_{cd0}{}^2-24\,\mu_{cd0}-15\right)\epsilon_w{}^2+8\,\mu_{cd0}\,\epsilon_s{}^2\right)\mu_{dh0}{}^4\mu_{de0} \\
-4\,\mu_{dh0}{}^6\epsilon_s{}^6
\end{array}\right)\end{array}}{\left(\begin{array}{c}
2\,\epsilon_w{}^2\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\left(\mu_{cd0}+1\right)^4\mu_{de0}{}^3 \\
-4\left(\begin{array}{c}-1/2\,\mu_{cd0}{}^2\epsilon_s{}^4 \\ +\epsilon_w{}^2\left(\mu_{cd0}+1\right)\left(\mu_{cd0}{}^2+4\,\mu_{cd0}+2\right)\epsilon_s{}^2 \\ -\epsilon_w{}^4\left(\mu_{cd0}+5/2\right)\left(\mu_{cd0}+1\right)^2\end{array}\right)\mu_{dh0}{}^2\epsilon_s{}^2\mu_{de0}{}^2 \\
+2\,\mu_{dh0}{}^4\left(-2\,\mu_{cd0}\,\epsilon_s{}^2+\epsilon_w{}^2\left(\mu_{cd0}{}^2+6\,\mu_{cd0}+6\right)\right)\epsilon_s{}^4\mu_{de0}+2\,\mu_{dh0}{}^6\epsilon_s{}^6
\end{array}\right)}}{\left(N_4\right)_{\epsilon_d}}. \tag{24}
$$

Equation (24) is accommodated ϵ_w, which has been extracted using the H_2 optimisation method. In order to perform the optimisation procedure, Eq. (24) is substituted in the third expression of Eq. (20). Hence, the closed-form expression for the optimal ϵ_w is derived as

$$
\left(\epsilon_w\right)_{\rm opt} = \sqrt{\frac{\left(\begin{array}{c}7\,\mu_{cd0}{}^5\mu_{de0}-7\,\mu_{cd0}{}^4\mu_{dh0}{}^2-16\,\mu_{cd0}{}^4\mu_{de0}+16\,\mu_{cd0}{}^3\mu_{dh0}{}^2 \\ -35\,\mu_{cd0}{}^3\mu_{de0}+37\,\mu_{cd0}{}^2\mu_{dh0}{}^2-20\,\mu_{cd0}{}^2\mu_{de0}+12\,\mu_{cd0}\,\mu_{dh0}{}^2 \\ -8\,\mu_{cd0}\,\mu_{de0}+8\,\mu_{dh0}{}^2\end{array}\right)\epsilon_s{}^2}{7\,\mu_{de0}\left(4-\mu_{cd0}\right)\left(\mu_{cd0}+1\right)^2\mu_{cd0}{}^2}}. \tag{25}
$$

The real root with the positive values is considered from the Eq. (25) to evaluate the optimal frequency of the liquid. The frequency ratio of the liquid inside the container has been derived using Eq. (25) and expressed as

$$(\eta_w)_{\text{opt}} = \frac{(\epsilon_w)_{\text{opt}}}{\epsilon_s} = \sqrt{\frac{\begin{pmatrix} 7\,\mu_{cd0}{}^5\mu_{de0} - 7\,\mu_{cd0}{}^4\mu_{dh0}{}^2 - 16\,\mu_{cd0}{}^4\mu_{de0} \\ +16\,\mu_{cd0}{}^3\mu_{dh0}{}^2 - 35\,\mu_{cd0}{}^3\mu_{de0} + 37\,\mu_{cd0}{}^2\mu_{dh0}{}^2 \\ -20\,\mu_{cd0}{}^2\mu_{de0} + 12\,\mu_{cd0}\,\mu_{dh0}{}^2 - 8\,\mu_{cd0}\,\mu_{de0} \\ +8\,\mu_{dh0}{}^2 \end{pmatrix}}{7\,\mu_{de0}\,(4-\mu_{cd0})\,(\mu_{cd0}+1)^2\,\mu_{cd0}{}^2}}. \tag{26}$$

Equation (25) is substituted in Eq. (23) to derive the optimal natural frequency of the container with liquid/damper in the matter of closed-form expression and expressed as

$$(\epsilon_d)^2_{\text{opt}} = \frac{\begin{pmatrix} 14\,\mu_{cd0}{}^4\mu_{dh0}{}^2 - 14\,\mu_{cd0}{}^5\mu_{de0} + 30\,\mu_{cd0}{}^4\mu_{de0} - 30\,\mu_{cd0}{}^3\mu_{dh0}{}^2 \\ +84\,\mu_{cd0}{}^3\mu_{de0} - 86\,\mu_{cd0}{}^2\mu_{dh0}{}^2 + 48\,\mu_{cd0}{}^2\mu_{de0} - 40\,\mu_{cd0}\,\mu_{dh0}{}^2 \\ +8\,\mu_{cd0}\,\mu_{de0} - 8\,\mu_{dh0}{}^2 \end{pmatrix}\begin{pmatrix} 7\,\mu_{cd0}{}^6\mu_{de0} - 7\,\mu_{dh0}{}^2\mu_{cd0}{}^5 - 29\,\mu_{cd0}{}^5\mu_{de0} + 29\,\mu_{cd0}{}^4\mu_{dh0}{}^2 \\ -12\,\mu_{cd0}{}^4\mu_{de0} + 13\,\mu_{cd0}{}^3\mu_{dh0}{}^2 + 60\,\mu_{cd0}{}^3\mu_{de0} - 73\,\mu_{cd0}{}^2\mu_{dh0}{}^2 \\ +44\,\mu_{cd0}{}^2\mu_{de0} - 8\,\mu_{cd0}\,\mu_{dh0}{}^2 + 8\,\mu_{cd0}\,\mu_{de0} - 8\,\mu_{dh0}{}^2 \end{pmatrix}\epsilon_s^2\begin{pmatrix} 7\,\mu_{cd0}{}^5\mu_{de0} - 7\,\mu_{cd0}{}^4\mu_{dh0}{}^2 - 16\,\mu_{cd0}{}^4\mu_{de0} + 16\,\mu_{cd0}{}^3\mu_{dh0}{}^2 \\ -35\,\mu_{cd0}{}^3\mu_{de0} + 37\,\mu_{cd0}{}^2\mu_{dh0}{}^2 - 20\,\mu_{cd0}{}^2\mu_{de0} + 12\,\mu_{cd0}\,\mu_{dh0}{}^2 \\ -8\,\mu_{cd0}\,\mu_{de0} + 8\,\mu_{dh0}{}^2 \end{pmatrix}}{(\mu_{cd0}+1)^2\begin{pmatrix} 1372\,\mu_{cd0}{}^{15}\mu_{de0}{}^3 - 4116\,\mu_{cd0}{}^{14}\mu_{de0}{}^2\mu_{dh0}{}^2 + 4116\,\mu_{cd0}{}^{13}\mu_{de0}\,\mu_{dh0}{}^4 \\ -1372\,\mu_{cd0}{}^{12}\mu_{dh0}{}^6 - 9016\,\mu_{cd0}{}^{14}\mu_{de0}{}^3 + 25676\,\mu_{cd0}{}^{13}\mu_{de0}{}^2\mu_{dh0}{}^2 \\ -24304\,\mu_{cd0}{}^{12}\mu_{de0}\,\mu_{dh0}{}^4 + 7644\,\mu_{cd0}{}^{11}\mu_{dh0}{}^6 - 3584\,\mu_{cd0}{}^{13}\mu_{de0}{}^3 \\ +25648\,\mu_{cd0}{}^{12}\mu_{de0}{}^2\mu_{dh0}{}^2 - 40544\,\mu_{cd0}{}^{11}\mu_{de0}\,\mu_{dh0}{}^4 + 74584\,\mu_{cd0}{}^{12}\mu_{de0}{}^3 \\ -268111\,\mu_{cd0}{}^{11}\mu_{de0}{}^2\mu_{dh0}{}^2 - 118551\,\mu_{cd0}{}^9\mu_{dh0}{}^6 + 74900\,\mu_{cd0}{}^{11}\mu_{de0}{}^3 \\ -240174\,\mu_{cd0}{}^{10}\mu_{de0}{}^2\mu_{dh0}{}^2 + 260891\,\mu_{cd0}{}^9\mu_{de0}\,\mu_{dh0}{}^4 - 95617\,\mu_{cd0}{}^8\mu_{dh0}{}^6 \\ -189184\,\mu_{cd0}{}^{10}\mu_{de0}{}^3 + 739901\,\mu_{cd0}{}^9\mu_{de0}{}^2\mu_{dh0}{}^2 - 935990\,\mu_{cd0}{}^8\mu_{de0}\,\mu_{dh0}{}^4 \\ -513392\,\mu_{cd0}{}^9\mu_{de0}{}^3 + 1601052\,\mu_{cd0}{}^8\mu_{de0}{}^2\mu_{dh0}{}^2 + 541457\,\mu_{cd0}{}^6\mu_{dh0}{}^6 \\ -575840\,\mu_{cd0}{}^8\mu_{de0}{}^3 + 1513216\,\mu_{cd0}{}^7\mu_{de0}{}^2\mu_{dh0}{}^2 - 1269436\,\mu_{cd0}{}^6\mu_{de0}\,\mu_{dh0}{}^4 \\ +328772\,\mu_{cd0}{}^5\mu_{dh0}{}^6 - 388096\,\mu_{cd0}{}^7\mu_{de0}{}^3 + 906416\,\mu_{cd0}{}^6\mu_{de0}{}^2\mu_{dh0}{}^2 \\ -709272\,\mu_{cd0}{}^5\mu_{de0}\,\mu_{dh0}{}^4 - 169216\,\mu_{cd0}{}^6\mu_{de0}{}^3 + 360640\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2 \\ +54336\,\mu_{cd0}{}^3\mu_{dh0}{}^6 - 46784\,\mu_{cd0}{}^5\mu_{de0}{}^3 + 98112\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2 \\ -70848\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4 + 19520\,\mu_{cd0}{}^2\mu_{dh0}{}^6 - 7424\,\mu_{cd0}{}^4\mu_{de0}{}^3 \\ +17152\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2 - 12032\,\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^4 + 2304\,\mu_{cd0}\,\mu_{dh0}{}^6 \\ -512\,\mu_{cd0}{}^3\mu_{de0}{}^3 + 1536\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2 - 1536\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^4 \\ +512\,\mu_{dh0}{}^6 - 1628619\,\mu_{cd0}{}^7\mu_{de0}\,\mu_{dh0}{}^4 + 312078\,\mu_{cd0}{}^{10}\mu_{de0}\,\mu_{dh0}{}^4 \\ +18480\,\mu_{cd0}{}^{10}\mu_{dh0}{}^6 + 385245\,\mu_{cd0}{}^7\mu_{dh0}{}^6 + 200520\,\mu_{cd0}{}^4\mu_{dh0}{}^6 \\ -256128\,\mu_{cd0}{}^4\mu_{de0}\,\mu_{dh0}{}^4 \end{pmatrix}}. \tag{27}$$

Equation (25) and Eq. (60) are substituted in Eq. (21) to derive the optimal damping ratio of the container with liquid/damper in the matter of closed-form expression and expressed as

$$(\xi_d)_{\text{opt}} = \sqrt{\frac{\begin{pmatrix} 7\,\mu_{cd0}{}^5\mu_{de0} - 7\,\mu_{cd0}{}^4\mu_{dh0}{}^2 - 15\,\mu_{cd0}{}^4\mu_{de0} + 15\,\mu_{cd0}{}^3\mu_{dh0}{}^2 \\ -42\,\mu_{cd0}{}^3\mu_{de0} + 43\,\mu_{cd0}{}^2\mu_{dh0}{}^2 - 24\,\mu_{cd0}{}^2\mu_{de0} + 20\,\mu_{cd0}\,\mu_{dh0}{}^2 \\ -4\,\mu_{cd0}\,\mu_{de0} + 4\,\mu_{dh0}{}^2 \end{pmatrix} N_5\,(\mu_{cd0}-4)\,\mu_{cd0}}{(2\,\mu_{cd0}+2)\,N_6 N_7\begin{pmatrix} 196\,\mu_{cd0}{}^{10}\mu_{de0}{}^2 - 392\,\mu_{cd0}{}^9\mu_{de0}\,\mu_{dh0}{}^2 + 196\,\mu_{cd0}{}^8\mu_{dh0}{}^4 \\ -840\,\mu_{cd0}{}^9\mu_{de0}{}^2 + 1680\,\mu_{cd0}{}^8\mu_{de0}\,\mu_{dh0}{}^2 - 840\,\mu_{cd0}{}^7\mu_{dh0}{}^4 \\ -1452\,\mu_{cd0}{}^8\mu_{de0}{}^2 + 3205\,\mu_{cd0}{}^7\mu_{de0}\,\mu_{dh0}{}^2 - 1753\,\mu_{cd0}{}^6\mu_{dh0}{}^4 \\ +3696\,\mu_{cd0}{}^7\mu_{de0}{}^2 - 9283\,\mu_{cd0}{}^6\mu_{de0}\,\mu_{dh0}{}^2 + 5636\,\mu_{cd0}{}^5\mu_{dh0}{}^4 \\ +9712\,\mu_{cd0}{}^6\mu_{de0}{}^2 - 18384\,\mu_{cd0}{}^5\mu_{de0}\,\mu_{dh0}{}^2 + 8361\,\mu_{cd0}{}^4\mu_{dh0}{}^4 \\ +8544\,\mu_{cd0}{}^5\mu_{de0}{}^2 - 11120\,\mu_{cd0}{}^4\mu_{de0}\,\mu_{dh0}{}^2 + 2712\,\mu_{cd0}{}^3\mu_{dh0}{}^4 \\ +3648\,\mu_{cd0}{}^4\mu_{de0}{}^2 - 3984\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^2 + 1632\,\mu_{cd0}{}^2\mu_{dh0}{}^4 \\ +768\,\mu_{cd0}{}^3\mu_{de0}{}^2 - 960\,\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^2 + 192\,\mu_{cd0}\,\mu_{dh0}{}^4 \\ +64\,\mu_{de0}{}^2\mu_{cd0}{}^2 - 128\,\mu_{dh0}{}^2\mu_{de0}\,\mu_{cd0} + 64\,\mu_{dh0}{}^4 \end{pmatrix}}}. \tag{28}$$

The closed-form expression for N_5, N_6, and N_7 from Eq. (28) is listed in Appendix A. The standard deviation of the velocity of the liquid inside the container is derived as

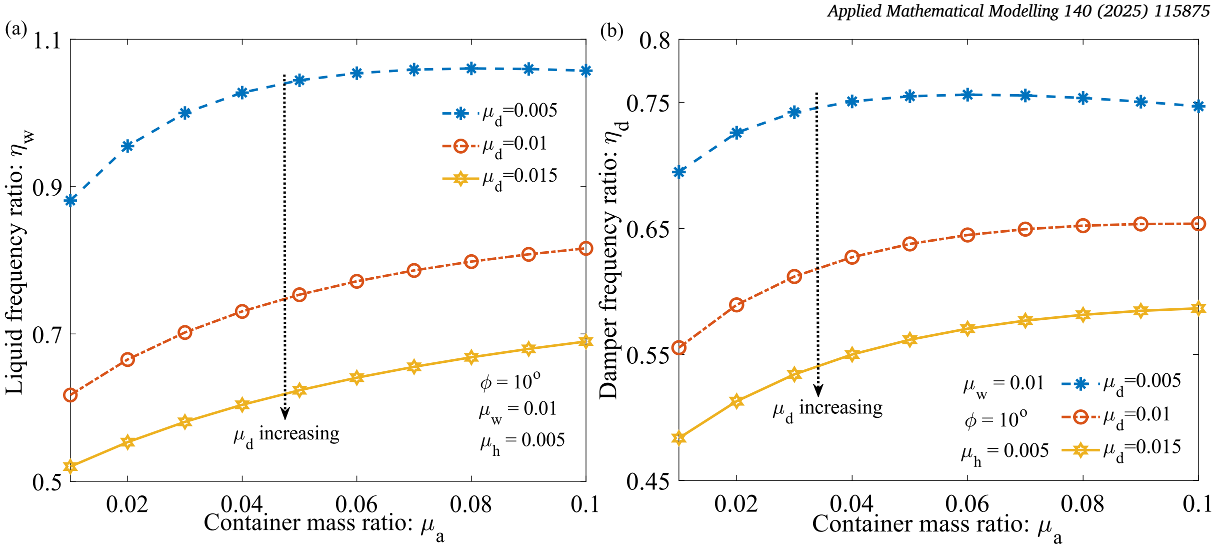

Fig. 2. (a) Liquid's and (b) damper's frequency ratios are evaluated by considering them as a function of container mass ratio. The amplifier's mass ratio is varied, i.e. $\mu_d = 0.005$, $\mu_d = 0.005$, and $\mu_d = 0.015$. The amplifier's angle and mass ratios of the liquid inside the horizontal and total length of the container are taken constant, i.e. $\phi = 10^o$, $\mu_w = 0.01$, and $\mu_h = 0.005$.

$$\sigma^2_{\dot{w}_d} = \frac{\mu_{cd0}\,\epsilon_d\,\pi\,S_0 \left(\begin{array}{c} 4\,\mu_{cd0}{}^2\mu_{de0}{}^2\xi_d{}^2\epsilon_s{}^4\epsilon_w{}^2 - 8\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2\xi_d{}^2\epsilon_s{}^4\epsilon_w{}^2 \\ +4\,\mu_{dh0}{}^4\xi_d{}^2\epsilon_s{}^4\epsilon_w{}^2 + 4\,\mu_{de0}\,\mu_{dh0}{}^2\xi_d{}^2\epsilon_s{}^2\epsilon_w{}^4 \\ +\mu_{cd0}{}^2\mu_{de0}{}^2\epsilon_d{}^2\epsilon_s{}^4 - 2\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4 \\ -2\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^2 + \mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_w{}^4 \\ +\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^4 + 2\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^2 + \mu_{de0}\,\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_w{}^4 \end{array} \right)}{2\left(\mu_{cd0}\,\mu_{de0}\,\epsilon_s{}^2 - \mu_{cd0}\,\mu_{de0}\,\epsilon_w{}^2 - \mu_{dh0}{}^2\epsilon_s{}^2\right)^2 \xi_d\,\mu_{de0}\,\epsilon_w{}^4}. \tag{29}$$

Equation (25), Eq. (60), Eq. (28), and Eq. (29) are substituted in the second expression of Eq. (8) to derive the optimal damping ratio of the liquid inside the container and expressed as

$$(\xi_e)_{\text{opt}} = \frac{\left(\sigma_{\dot{w}_d}\right)_{(\epsilon_d,\xi_d,\epsilon_w)_{\text{opt}}} \xi_w}{\sqrt{2\pi}}. \tag{30}$$

A parametric study has been performed by using the above-derived optimal closed-form solutions. First, the frequency ratio variations as a function of container mass ratio are graphically obtained and presented in Fig. 2. Specifically, Fig. 2 (a) addresses the changes in the liquid's frequency ratio when the mass of the container varies from 0.01 to 0.10. Equation (26) is employed to obtain this graph. According to this graph, the liquid frequency ratio increases as the container mass ratio increases. However, the characteristics are opposite when the mass of the amplifier increases. The liquid frequency ratio decreases as the amplifier mass ratio increases. The liquid frequency ratio needs to be maintained high to achieve robust vibration reduction from the nonlinear inertial amplifier liquid column dampers. Therefore, a moderate container mass ratio and lower amplifier ratio are endorsed to design an optimum nonlinear inertia amplifier liquid column damper. The same characteristics are also found in the damper frequency ratio graph, which is shown in Fig. 2 (b). In this graph, the damper frequency ratio is derived by making the container mass ratio as its function. The differences have been observed while changing the values of the container mass ratio from 0.01 to 0.1. More specifically, the damper frequency ratio increases with the increment of the container mass ratio, and the damper frequency ratio is decreased when the amplifier mass ratio is increased. The very low-frequency ratio provides extreme flexibility to the damper's base layer. As a result, the damper will be damaged by the high amplitude vibrations. Therefore, a moderate frequency ratio is suitable for an inertially amplified liquid column damper. To achieve that, a moderate container mass ratio and a lower amplifier mass ratio are suggested.

The damping ratio variations are investigated, like the frequency ratio variations using Eq. (28) and Eq. (30). Therefore, the differences in the optimal damping ratios of the liquid are obtained using container mass ratio as its function and graphically represented in Fig. 3 (a). According to the graph, the liquid damping ratio decreases as the container mass ratio increases. In contrast, the liquid damping ratio increases with the increment of the amplifier mass ratio. However, this decremental damping ratio is appropriate for the proposed liquid column damper and cost-effective. Therefore, a moderate container mass ratio and a lower amplifier mass ratio are suggested to achieve a cost-effective design for the proposed liquid column damper. The lower damping ratios, like 0.35 to 0.4 ranges, are sufficient for these types of novel dampers. Otherwise, the dynamic system will become over-damped and may be damaged during long-period vibrations. The opposite characteristics are observed for the damping ratio of the damper. The graphical representation of the damper damping ratio variations is obtained by varying the container mass ratio, which is presented in its optimal closed-form solutions and presented in Fig. 3 (b). According to this graph, the damping ratio increases as the container mass

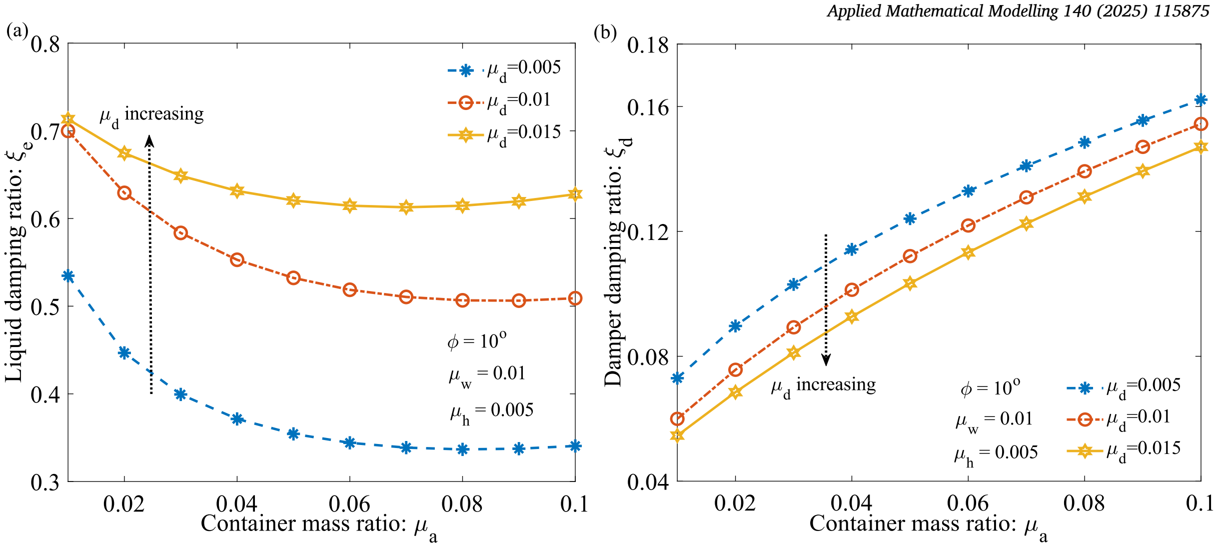

Fig. 3. The variations of the optimal damping ratios of the (a) liquid and the (b) damper are obtained using Eq. (28) and Eq. (30). The container mass ratio and the amplifier's mass ratio are varied, i.e. $\mu_d = 0.005$, $\mu_d = 0.005$, and $\mu_d = 0.015$. The angle and mass ratios of the liquid inside the amplifier's container are kept constant, as well as the entire length of the container, i.e. $\phi = 10^o$, $\mu_w = 0.01$, and $\mu_h = 0.005$.

ratio increases. In contrast, the damper damping ratio decreases as the amplifier mass ratio increases. The over-lowered damping ratio makes the damper under-damped, which can not generate sufficient inertial force in the system when the base loading starts applying. Therefore, a moderate damping ratio is required, i.e., in the 0.05 to 0.15 ranges, to achieve robust performance from the liquid column damper. All-inclusive, the moderate container mass ratio, a small value of amplifier mass ratio is required to design optimum nonlinear inertial amplifier liquid column damper.

3.2. H_∞ optimisation

It has been considered that the controlled single-degree-of-freedom system is subjected to harmonic excitation. This consideration allows us to pursue further the analytical optimisation procedure using the fixed point theory/H_∞ optimisation method. The optimal design parameters for this novel damper have been achieved using the H_∞ optimisation method, and this optimisation approach provides additional benefits to the dampers while designing the damper for achieving robust vibration reduction capacity from it. The accuracy and robustness of its vibration reduction performances may increase during calibration. To apply this optimisation approach, it is considered that the controlled SDOF system is subjected to the harmonic base excitation, and Eq. (14) has been non-dimensionalised. The non-dimensionalised version of the frequency response function has been derived as

$$\begin{bmatrix} -\eta^2 + 2\,\mathrm{i}\xi_s\,\eta + 1 & A_{12} & 0 \\ -\mu_{cd0}\,\eta^2 & A_{22} & -\mu_{dh0}\,\eta^2 \\ -\frac{\mu_{dh0}\,\eta^2}{\mu_{de0}} & -\frac{\mu_{dh0}\,\eta^2}{\mu_{de0}} & -\eta^2 + {\eta_w}^2 \end{bmatrix} \begin{Bmatrix} V_s \\ V_d \\ W_d \end{Bmatrix} = - \begin{bmatrix} 1 \\ \mu_{cd0} \\ \frac{\mu_{dh0}}{\mu_{de0}} \end{bmatrix} \frac{V_g}{\epsilon_s^2}, \tag{31}$$

$$A_{12} = -2\,\mathrm{i}\mu_{cd0}\,\xi_d\,\eta_d\,\eta - \mu_{cd0}\,{\eta_d}^2, \quad \text{and} \quad A_{22} = -\mu_{cd0}\,\eta^2 + 2\,\mathrm{i}\mu_{cd0}\,\xi_d\,\eta_d\,\eta + \mu_{cd0}\,{\eta_d}^2.$$

The dynamic response of the single degree of freedom system is derived using Eq. (31) and expressed as

$$\tilde{H}_s = \left(\frac{V_s}{V_g}\right)\epsilon_s^2 = \frac{\begin{matrix} \eta^2{\eta_d}^2{\mu_{cd0}}^2\mu_{de0} - \eta^2{\eta_d}^2\mu_{cd0}\,{\mu_{dh0}}^2 - {\eta_d}^2{\eta_w}^2{\mu_{cd0}}^2\mu_{de0} \\ -\eta^4\mu_{cd0}\,\mu_{de0} + {\mu_{dh0}}^2\eta^4 + \eta^2{\eta_d}^2\mu_{cd0}\,\mu_{de0} \\ +\eta^2{\eta_w}^2\mu_{cd0}\,\mu_{de0} - {\eta_d}^2{\eta_w}^2\mu_{cd0}\,\mu_{de0} \\ +\mathrm{i}\left(\begin{matrix} 2\,\eta^3\eta_d\,{\mu_{cd0}}^2\mu_{de0}\,\xi_d - 2\,\eta^3\eta_d\,\mu_{cd0}\,{\mu_{dh0}}^2\xi_d \\ -2\,\eta\,\eta_d\,{\eta_w}^2{\mu_{cd0}}^2\mu_{de0}\,\xi_d + 2\,\eta^3\eta_d\,\mu_{cd0}\,\mu_{de0}\,\xi_d \\ -2\,\eta\,\eta_d\,{\eta_w}^2\mu_{cd0}\,\mu_{de0}\,\xi_d \end{matrix}\right) \end{matrix}}{\tilde{\delta}_d}. \tag{32}$$

In the same way, the dynamic response of the lateral deflection of the container with liquid mass is derived and expressed as

$$\tilde{H}_d = \left(\frac{V_d}{V_g}\right)\epsilon_s^2 = \frac{\left(\left(2\,\mathrm{i}\xi_s\,\eta + 1\right)\left(\eta^2\mu_{cd0}\,\mu_{de0} - \eta^2{\mu_{dh0}}^2 - {\eta_w}^2\mu_{cd0}\,\mu_{de0}\right)\right)}{\tilde{\delta}_d}. \tag{33}$$

The dynamic response of the vertical deflection of the liquid mass is derived as

$$\tilde{H}_{wd} = \left(\frac{W_d}{V_g}\right)\epsilon_s^2 = \frac{\begin{pmatrix} 4\,\eta_d\,\mu_{dh0}\,\mu_{cd0}\,\eta^2\xi_d\,\xi_s - \eta_d{}^2\mu_{dh0}\,\mu_{cd0} \\ +\mathrm{i}\left(-2\,\eta_d{}^2\mu_{dh0}\,\mu_{cd0}\,\eta\,\xi_s - 2\,\eta_d\,\mu_{dh0}\,\mu_{cd0}\,\eta\,\xi_d\right)\end{pmatrix}}{\tilde{\delta}_d}. \tag{34}$$

The denominator of Eq. (32), Eq. (33), and Eq. (34) are derived as

$$\tilde{\delta}_d = \begin{matrix} \eta^4\eta_d{}^2\mu_{cd0}{}^2\mu_{de0} - \eta^4\eta_d{}^2\mu_{cd0}\,\mu_{dh0}{}^2 + 4\,\eta^4\eta_d\,\mu_{cd0}\,\mu_{de0}\,\xi_d\,\xi_s + \eta^6\mu_{dh0}{}^2 \\ -\eta^2\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0} - 4\,\eta^2\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\xi_d\,\xi_s - \eta^6\mu_{cd0}\,\mu_{de0} - \mu_{dh0}{}^2\eta^4 \\ +\eta^4\eta_d{}^2\mu_{cd0}\,\mu_{de0} + \eta^4\eta_w{}^2\mu_{cd0}\,\mu_{de0} - \eta^2\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} + \eta^4\mu_{cd0}\,\mu_{de0} \\ -\eta^2\eta_d{}^2\mu_{cd0}\,\mu_{de0} - \eta^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} + \eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} \\ +\mathrm{i}\begin{pmatrix} 2\,\eta^5\eta_d\,\mu_{cd0}{}^2\mu_{de0}\,\xi_d - 2\,\eta^5\eta_d\,\mu_{cd0}\,\mu_{dh0}{}^2\xi_d - 2\,\eta^3\eta_d\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}\,\xi_d \\ +2\,\eta^5\eta_d\,\mu_{cd0}\,\mu_{de0}\,\xi_d - 2\,\eta^3\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\xi_d + 2\,\eta^5\mu_{cd0}\,\mu_{de0}\,\xi_s \\ -2\,\eta^5\mu_{dh0}{}^2\xi_s - 2\,\eta^3\eta_d{}^2\mu_{cd0}\,\mu_{de0}\,\xi_s - 2\,\eta^3\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\xi_s \\ +2\,\eta\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\xi_s - 2\,\eta^3\eta_d\,\mu_{cd0}\,\mu_{de0}\,\xi_d + 2\,\eta\,\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\xi_d \end{pmatrix} \end{matrix}. \tag{35}$$

Once the starting circumstances have been taken into consideration, such as the fact that the damping ratio of the system with a single degree of freedom is equal to zero, the consequent of Eq. (32) and Eq. (35) is expressed as

$$|\tilde{H}_s(\eta)| = \sqrt{\frac{H_1^2 + \xi_d^2 H_2^2}{H_3^2 + \xi_d^2 H_4^2}} = \left|\frac{H_2}{H_4}\right|\sqrt{\frac{\frac{H_1^2}{H_2^2} + \xi_d^2}{\frac{H_3^2}{H_4^2} + \xi_d^2}}. \tag{36}$$

Equation (36) is employed to derive two constraints and expressed as

$$\left.\frac{H_1}{H_2}\right|_{\eta_j} = \left.\frac{H_3}{H_4}\right|_{\eta_j} \quad \text{and} \quad \left|\frac{H_2}{H_4}\right|_{\eta_1} = \left|\frac{H_2}{H_4}\right|_{\eta_2}. \tag{37}$$

The closed-form expressions for H_1, H_2, H_3, and H_4 are derived and listed below.

$$H_1 = \begin{matrix} \eta^2\eta_d{}^2\mu_{cd0}{}^2\mu_{de0} - \eta^2\eta_d{}^2\mu_{cd0}\,\mu_{dh0}{}^2 - \eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0} - \eta^4\mu_{cd0}\,\mu_{de0} \\ +\eta^4\mu_{dh0}{}^2 + \eta^2\eta_d{}^2\mu_{cd0}\,\mu_{de0} + \eta^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} - \eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} \end{matrix}. \tag{38}$$

$$H_2 = \begin{matrix} 2\,\eta^3\eta_d\,\mu_{cd0}{}^2\mu_{de0} - 2\,\eta^3\eta_d\,\mu_{cd0}\,\mu_{dh0}{}^2 - 2\,\eta\,\eta_d\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0} + 2\,\eta^3\eta_d\,\mu_{cd0}\,\mu_{de0} \\ -2\,\eta\,\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0} \end{matrix}. \tag{39}$$

$$H_3 = \begin{matrix} \eta^4\eta_d{}^2\mu_{cd0}{}^2\mu_{de0} - \eta^4\eta_d{}^2\mu_{cd0}\,\mu_{dh0}{}^2 - \eta^2\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0} - \eta^6\mu_{cd0}\,\mu_{de0} \\ +\eta^4\eta_d{}^2\mu_{cd0}\,\mu_{de0} + \eta^4\eta_w{}^2\mu_{cd0}\,\mu_{de0} - \eta^2\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} + \eta^4\mu_{cd0}\,\mu_{de0} \\ -\eta^4\mu_{dh0}{}^2 - \eta^2\eta_d{}^2\mu_{cd0}\,\mu_{de0} - \eta^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} + \eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0} + \eta^6\mu_{dh0}{}^2 \end{matrix}. \tag{40}$$

$$H_4 = \begin{matrix} 2\,\eta^5\eta_d\,\mu_{cd0}{}^2\mu_{de0} - 2\,\eta^5\eta_d\,\mu_{cd0}\,\mu_{dh0}{}^2 - 2\,\eta^3\eta_d\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0} \\ +2\,\eta^5\eta_d\,\mu_{cd0}\,\mu_{de0} - 2\,\eta^3\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0} - 2\,\eta^3\eta_d\,\mu_{cd0}\,\mu_{de0} \\ +2\,\eta\,\eta_d\,\eta_w{}^2\mu_{cd0}\,\mu_{de0} \end{matrix}. \tag{41}$$

A closed-form expression is generated using the first expression of Eq. (37) and expressed as

$$\begin{matrix} \begin{pmatrix} 2\,\mu_{cd0}{}^2\mu_{de0}{}^2 - 4\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 + 2\,\mu_{dh0}{}^4 \\ +2\,\mu_{cd0}\,\mu_{de0}{}^2 - 2\,\mu_{de0}\,\mu_{dh0}{}^2 \end{pmatrix}\eta^8 \\ +\begin{pmatrix} -2\,\eta_d{}^2\mu_{cd0}{}^3\mu_{de0}{}^2 + 4\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^2 - 2\,\eta_d{}^2\mu_{cd0}\,\mu_{dh0}{}^4 \\ -4\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 + 4\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 - 4\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 \\ +4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 - 2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}{}^2 - 4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2 \\ +2\,\eta_w{}^2\mu_{de0}\,\mu_{dh0}{}^2 - \mu_{cd0}{}^2\mu_{de0}{}^2 + 2\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 - \mu_{dh0}{}^4 \\ -2\,\mu_{cd0}\,\mu_{de0}{}^2 + 2\,\mu_{de0}\,\mu_{dh0}{}^2 \end{pmatrix}\eta^6 \\ +\begin{pmatrix} 4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^3\mu_{de0}{}^2 - 4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^2 + 8\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 \\ -4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 + 2\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2 + 4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2 \\ +2\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2 + 2\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 - 2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 \\ +2\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 - 2\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 + 2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}{}^2 \\ +4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2 - 2\,\eta_w{}^2\mu_{de0}\,\mu_{dh0}{}^2 \end{pmatrix}\eta^4 \\ +\begin{pmatrix} -2\,\eta_d{}^2\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}{}^2 - 4\,\eta_d{}^2\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2 - 2\,\eta_d{}^2\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2 \\ -4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2 + 2\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2 - \eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2 \\ -4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2 - 2\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2 \\ +2\,\eta_d{}^2\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2 + 2\,\eta_d{}^2\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2 \end{pmatrix}\eta^2 \end{matrix} = 0. \tag{42}$$

Equation (42) has been re-written as

$$\begin{aligned}&\eta^8+\left(-\eta_1{}^2-\eta_2{}^2-\eta_3{}^2-\eta_4{}^2\right)\eta^6\\&+\left(\eta_1{}^2\eta_2{}^2+\eta_1{}^2\eta_3{}^2+\eta_1{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2+\eta_2{}^2\eta_4{}^2+\eta_3{}^2\eta_4{}^2\right)\eta^4\\&+\left(-\eta_1{}^2\eta_2{}^2\eta_3{}^2-\eta_1{}^2\eta_4{}^2\eta_2{}^2-\eta_1{}^2\eta_3{}^2\eta_4{}^2-\eta_2{}^2\eta_3{}^2\eta_4{}^2\right)\eta^2+\eta_1{}^2\eta_2{}^2\eta_3{}^2\eta_4{}^2\end{aligned}=0. \tag{43}$$

Equation (42) and Eq. (43) are compared and the roots are derived as

$$\eta_1{}^2+\eta_2{}^2+\eta_3{}^2+\eta_4{}^2=\frac{\begin{array}{c}2\,\eta_d{}^2\mu_{cd0}{}^3\mu_{de0}{}^2-4\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^2+2\,\eta_d{}^2\mu_{cd0}\,\mu_{dh0}{}^4\\+4\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}{}^2-4\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+4\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2\\-4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}{}^2+4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2\\-2\,\eta_w{}^2\mu_{de0}\,\mu_{dh0}{}^2+\mu_{cd0}{}^2\mu_{de0}{}^2-2\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+\mu_{dh0}{}^4\\+2\,\mu_{cd0}\,\mu_{de0}{}^2-2\,\mu_{de0}\,\mu_{dh0}{}^2\end{array}}{\begin{array}{c}2\,\mu_{cd0}{}^2\mu_{de0}{}^2-4\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\mu_{dh0}{}^4+2\,\mu_{cd0}\,\mu_{de0}{}^2\\-2\,\mu_{de0}\,\mu_{dh0}{}^2\end{array}}. \tag{44}$$

$$\begin{aligned}&\eta_1{}^2\eta_2{}^2+\eta_1{}^2\eta_3{}^2+\eta_1{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2+\eta_2{}^2\eta_4{}^2+\eta_3{}^2\eta_4{}^2\\&=\frac{\begin{array}{c}4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^3\mu_{de0}{}^2-4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^2+8\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2\\-4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2+4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2\\+2\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2+2\,\eta_d{}^2\mu_{cd0}{}^2\mu_{de0}{}^2-2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2-2\,\eta_w{}^2\mu_{de0}\,\mu_{dh0}{}^2\\+2\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2-2\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}{}^2+4\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2\end{array}}{\begin{array}{c}2\,\mu_{cd0}{}^2\mu_{de0}{}^2-4\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\mu_{dh0}{}^4+2\,\mu_{cd0}\,\mu_{de0}{}^2\\-2\,\mu_{de0}\,\mu_{dh0}{}^2\end{array}}.\end{aligned} \tag{45}$$

$$\begin{aligned}&\eta_1{}^2\eta_2{}^2\eta_3{}^2+\eta_1{}^2\eta_2{}^2\eta_4{}^2+\eta_1{}^2\eta_3{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2\eta_4{}^2\\&=\frac{\eta_w{}^2\mu_{cd0}\,\mu_{de0}\left(\begin{array}{c}2\,\eta_d{}^2\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}+4\,\eta_d{}^2\eta_w{}^2\mu_{cd0}\,\mu_{de0}+2\,\eta_d{}^2\eta_w{}^2\mu_{de0}\\+4\,\eta_d{}^2\mu_{cd0}\,\mu_{de0}-2\,\eta_d{}^2\mu_{dh0}{}^2+\eta_w{}^2\mu_{cd0}\,\mu_{de0}+4\,\eta_d{}^2\mu_{de0}\\+2\,\eta_w{}^2\mu_{de0}\end{array}\right)}{2\,\mu_{cd0}{}^2\mu_{de0}{}^2-4\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^2+2\,\mu_{dh0}{}^4+2\,\mu_{cd0}\,\mu_{de0}{}^2-2\,\mu_{de0}\,\mu_{dh0}{}^2}.\end{aligned} \tag{46}$$

$$\eta_1{}^2\eta_2{}^2\eta_3{}^2\eta_4{}^2=\frac{\eta_d{}^2\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2\left(\mu_{cd0}+1\right)}{\left(\mu_{cd0}\,\mu_{de0}-\mu_{dh0}{}^2\right)\left(\mu_{cd0}\,\mu_{de0}-\mu_{dh0}{}^2+\mu_{de0}\right)}. \tag{47}$$

Another closed-form expression is generated using the second expression of Eq. (37) and expressed as

$$\eta_1^2+\eta_2^2=0 \quad \text{and} \quad \eta_1^2-\eta_2^2=0. \tag{48}$$

Equation (44), Eq. (45), Eq. (46), and Eq. (47) are written as

$$\begin{aligned}&\eta_1{}^2+\eta_2{}^2+\eta_3{}^2+\eta_4{}^2=U_1,\\&\eta_1{}^2\eta_2{}^2+\eta_1{}^2\eta_3{}^2+\eta_1{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2+\eta_2{}^2\eta_4{}^2+\eta_3{}^2\eta_4{}^2=U_2,\\&\eta_1{}^2\eta_2{}^2\eta_3{}^2+\eta_1{}^2\eta_2{}^2\eta_4{}^2+\eta_1{}^2\eta_3{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2\eta_4{}^2=U_3,\\&\eta_1{}^2\eta_2{}^2\eta_3{}^2\eta_4{}^2=U_4.\end{aligned} \tag{49}$$

The first expression of the Eq. (48) is substituted in the first expression of Eq. (49).

$$\eta_3{}^2+\eta_4{}^2=U_1. \tag{50}$$

The first expression of the Eq. (48) and Eq. (50) are substituted in the second expression of Eq. (49).

$$\begin{aligned}&\left(\eta_2{}^2+\eta_3{}^2+\eta_4{}^2\right)\eta_1{}^2+\left(\eta_3{}^2+\eta_4{}^2\right)\eta_2{}^2+\eta_3{}^2\eta_4{}^2=U_2,\\&\left(\eta_2{}^2+U_1\right)\eta_1{}^2+U_1\eta_2{}^2+\eta_3{}^2\eta_4{}^2=U_2,\\&\eta_1{}^2\eta_2{}^2+U_1\left(\eta_1{}^2+\eta_2{}^2\right)+\eta_3{}^2\eta_4{}^2=U_2,\\&\eta_1{}^2\eta_2{}^2+\eta_3{}^2\eta_4{}^2=U_2.\end{aligned} \tag{51}$$

The first expression of the Eq. (48) and Eq. (50) are substituted in the third expression of Eq. (49).

$$\begin{aligned}&\eta_1{}^2\eta_2{}^2\eta_3{}^2+\eta_1{}^2\eta_2{}^2\eta_4{}^2+\eta_1{}^2\eta_3{}^2\eta_4{}^2+\eta_2{}^2\eta_3{}^2\eta_4{}^2=U_3,\\&\eta_1{}^2\eta_2{}^2\left(\eta_3{}^2+\eta_4{}^2\right)+\eta_3{}^2\eta_4{}^2\left(\eta_1{}^2+\eta_2{}^2\right)=U_3,\\&\eta_1{}^2\eta_2{}^2=\frac{U_3}{\left(\eta_3{}^2+\eta_4{}^2\right)},\quad \text{or} \quad \eta_1{}^2\eta_2{}^2=\frac{U_3}{U_1}.\end{aligned} \tag{52}$$

The last expression of Eq. (52) is substituted in the fourth expression of Eq. (49) and the last expression of Eq. (51).

$$\eta_3^2\eta_4^2 = \frac{U_1U_4}{U_3} \quad \text{and} \quad \eta_3^2\eta_4^2 = \frac{\left(U_1U_2 - U_3\right)}{U_1}. \tag{53}$$

Equation (48) is substituted in the last expression of Eq. (52).

$$\eta_{1,2}^2 = \pm\sqrt{\frac{U_3}{U_1}} \quad \text{and} \quad \eta_{1,2}^2 = \pm\sqrt{\frac{\tilde{U}_3}{U_1}}. \tag{54}$$

$\tilde{U}_3$ has been derived as

$$\tilde{U}_3 = \frac{{\eta_w}^2\mu_{cd0}\,\mu_{de0}\left(\begin{array}{c} 2\,\mu_{de0}\,{\eta_d}^2{\eta_w}^2{\mu_{cd0}}^2 + 4\,\mu_{de0}\,{\eta_d}^2{\eta_w}^2\mu_{cd0} + 2\,\mu_{de0}\,{\eta_d}^2{\eta_w}^2 \\ +4\,\mu_{de0}\,{\eta_d}^2\mu_{cd0} - 2\,{\eta_d}^2{\mu_{dh0}}^2 + \mu_{de0}\,{\eta_w}^2\mu_{cd0} \\ +4\,\mu_{de0}\,{\eta_d}^2 + 2\,\mu_{de0}\,{\eta_w}^2 \end{array}\right)}{2\,\left({\mu_{dh0}}^2 - \mu_{cd0}\,\mu_{de0}\right)\left(\mu_{cd0}\,\mu_{de0} - {\mu_{dh0}}^2 + \mu_{de0}\right)}. \tag{55}$$

The first expression of Eq. (53) is rewritten as

$$\eta_3^2 = \frac{U_1U_2}{U_3\eta_4^2}. \tag{56}$$

Equation (56) is substituted in Eq. (50) to derive the roots for $\eta_{3,4}^2$.

$$\begin{aligned} &\frac{U_1U_2}{U_3\eta_4^2} + \eta_4^2 = U_1, \\ &\frac{U_1\,U_2}{U_3\,{\eta_4}^2} + {\eta_4}^2 - U_1 = 0, \\ &{\eta_4}^4U_3 - U_1\,U_3\,{\eta_4}^2 + U_1\,U_2 = 0, \\ &\eta_4^2 = \frac{U_1\,U_3 \pm \sqrt{{U_1}^2{U_3}^2 - 4\,U_3\,U_1\,U_2}}{2U_3}. \end{aligned} \tag{57}$$

The last expression of Eq. (57) is substituted in Eq. (50) to derive the closed-form expression for η_3^2.

$$\eta_3^2 = \frac{U_1\,U_3 \pm \sqrt{U_1\,U_3\,\left(U_1\,U_3 - 4\,U_2\right)}}{2U_3}. \tag{58}$$

The last expression of Eq. (52) and the first expression of Eq. (53) are substituted in the last expression of Eq. (51).

$$W_1\left(\eta_d^2\right)^3 + W_2\left(\eta_d^2\right)^2 + W_3\eta_d^2 + W_4 = 0. \tag{59}$$

where W_1, W_2, W_3, and W_4 are derived and listed in Appendix B. The exact closed-form expression for the optimal frequency ratio of the container with liquid has been derived using Eq. (59) and expressed as

$$\begin{aligned} &\left(\eta_{d1}\right)_{\text{opt}}^2 = \frac{-12\,W_3\,W_1 + 4\,{W_2}^2 - 2\,W_2W_5^{\frac{1}{3}} + W_5^{\frac{2}{3}}}{6W_1W_5^{\frac{1}{3}}}, \\ &\left(\eta_{d2}\right)_{\text{opt}}^2 = \frac{\begin{array}{c} \mathrm{i}\sqrt{3}{W_5}^{2/3} + 12\,\mathrm{i}\sqrt{3}W_1\,W_3 - 4\,\mathrm{i}\sqrt{3}{W_2}^2 - {W_5}^{2/3} \\ -4\,W_2\,\sqrt[3]{W_5} + 12\,W_3\,W_1 - 4\,{W_2}^2 \end{array}}{12W_1W_5^{\frac{1}{3}}}, \\ &\left(\eta_{d3}\right)_{\text{opt}}^2 = \frac{\begin{array}{c} -\mathrm{i}\sqrt{3}{W_5}^{2/3} - 12\,\mathrm{i}\sqrt{3}W_1\,W_3 + 4\,\mathrm{i}\sqrt{3}{W_2}^2 - {W_5}^{2/3} \\ -4\,W_2\,\sqrt[3]{W_5} + 12\,W_3\,W_1 - 4\,{W_2}^2 \end{array}}{12W_1W_5^{\frac{1}{3}}}. \end{aligned} \tag{60}$$

The real root with the positive values is considered from the Eq. (60) to evaluate the optimal frequency ratio of the damper. The exact closed-form expression for the optimal value of ξ_d has been derived by formulating a mathematical expression and expressed as

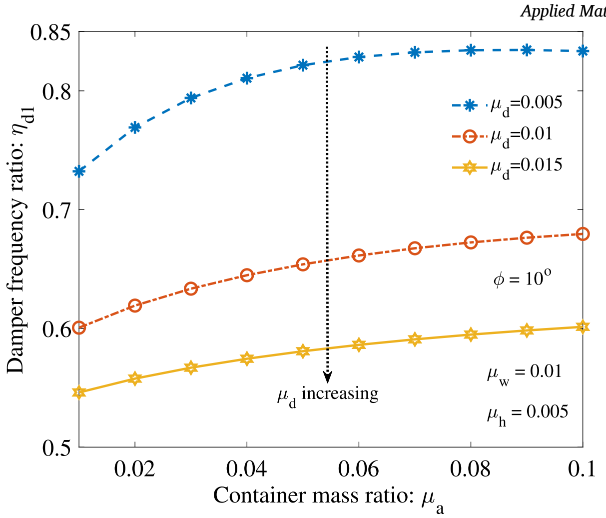

Fig. 4. The damper's frequency ratio as a function of container mass ratio. The amplifier's mass ratio is varied, i.e. $\mu_d = 0.005$, $\mu_d = 0.005$, and $\mu_d = 0.015$. Eq. (25) is applied in the Eq. (B.1), Eq. (B.2), Eq. (B.3), Eq. (B.4), Eq. (B.5), and the first expression of Eq. (60) to obtain this graph. The amplifier's angle and mass ratios of the liquid inside the horizontal and total length of the container are taken constant, i.e. $\phi = 10^o$, $\mu_w = 0.01$, and $\mu_h = 0.005$.

$$\left.\frac{\partial|\tilde{H}_s(\eta)|^2}{\partial\eta^2}\right|_{\eta^2_{1,2,3,4}} = 0 \quad \text{and} \quad \left(\xi_d\right)_{\text{opt}} = \sqrt{\frac{\xi^2_{d1} + \xi^2_{d2} + \xi^2_{d3} + \xi^2_{d4}}{4}}. \tag{61}$$

After applying Eq. (61), the closed-form expression for the optimal damping ratio of the container with liquid has been derived and expressed as

$$\begin{aligned} &Z_1 \left(\xi_d\right)^4_{\text{opt}} + Z_2 \left(\xi_d\right)^2_{\text{opt}} + Z_3 = 0, \\ &\left(\xi_d\right)^2_{\text{opt}} = \frac{-Z_2 \pm \sqrt{-4\,Z_1\,Z_3 + {Z_2}^2}}{2Z_1}. \end{aligned} \tag{62}$$

Z_1, Z_2, and Z_3 in Eq. (62) have been derived and listed in Appendix B The derived optimal closed-form solutions above have been used in a parametric investigation. First, as a function of container mass ratio, the frequency ratio fluctuations are computed graphically and shown in Fig. 4. In particular, as the mass of the container varies from 0.01 to 0.10, the changes in the frequency ratio of the liquid are addressed. Eq. (25) is applied in the Eq. (B.1), Eq. (B.2), Eq. (B.3), Eq. (B.4), Eq. (B.5), and the first expression of Eq. (60) to obtain this graph to produce this graph. This graph indicates that when the container mass ratio rises, the liquid frequency ratio rises as well. When the amplifier's mass increases, the characteristics change, nevertheless. As the amplifier mass ratio rises, the liquid frequency ratio falls. For the nonlinear inertial amplifier liquid column dampers to provide robust vibration reduction, a high moderate frequency ratio (near resonance frequency region) must be maintained, i.e. 0.7 to 0.99. Therefore, a moderate container mass ratio and a lower amplifier ratio are recommended to produce a superior nonlinear inertia amplifier liquid column damper. This graph is utilised further to generate optimum dynamic responses of the structures controlled by H_∞ optimised novel dampers. The same characteristics are observed in Fig. 5 (a) and Fig. 5 (b). The frequency ratio variations are graphically estimated as a function of container mass ratio and are displayed in Fig. 5 (a) and Fig. 5 (b). Specifically, the variations in the liquid's frequency ratio are discussed as the container's mass varies from 0.01 to 0.10. In order to generate these graphs, Eq. (25) is applied in the Eq. (B.1), Eq. (B.2), Eq. (B.3), Eq. (B.4), Eq. (B.5), and the second and third expressions of Eq. (60). This graph shows that the liquid frequency ratio increases in conjunction with the container mass ratio. However, the characteristics alter as the mass of the amplifier grows. The liquid frequency ratio decreases as the amplifier mass ratio increases. In order to achieve strong vibration reduction, the nonlinear inertial amplifier liquid column dampers need to be operated at a high moderate frequency ratio (near resonance frequency range), which is between 0.7 and 1.2 for the particular design of the liquid column dampers with higher resonating frequency regions. Therefore, a moderate container mass ratio and a smaller amplifier ratio are advised in order to create a superior nonlinear inertia amplifier liquid column damper. The optimum dynamic responses of the structures controlled by H_∞ optimised new dampers are further generated by utilising this graph.

Using Eq. (62), the damping ratio variations are examined similarly to the frequency ratio variations. As a result, the container mass ratio is used as the function to derive the variations in the optimal damping ratios of the damper, which are then graphically shown in Fig. 6. The graph indicates that when the container mass ratio rises, the damper damping ratio rises. This graph indicates that when the container mass ratio rises, the damping ratio rises as well. On the other hand, when the amplifier mass ratio rises, the damper damping ratio falls. When base loading begins to apply, the damper's under-damped state due to the excessively low damping ratio prevents it from producing enough inertial forces for the system. Consequently, to get a strong performance out of the H_∞ optimised liquid column damper, a moderate damping ratio—that is, 0.2 to 0.5 ranges—is needed. To construct the optimum

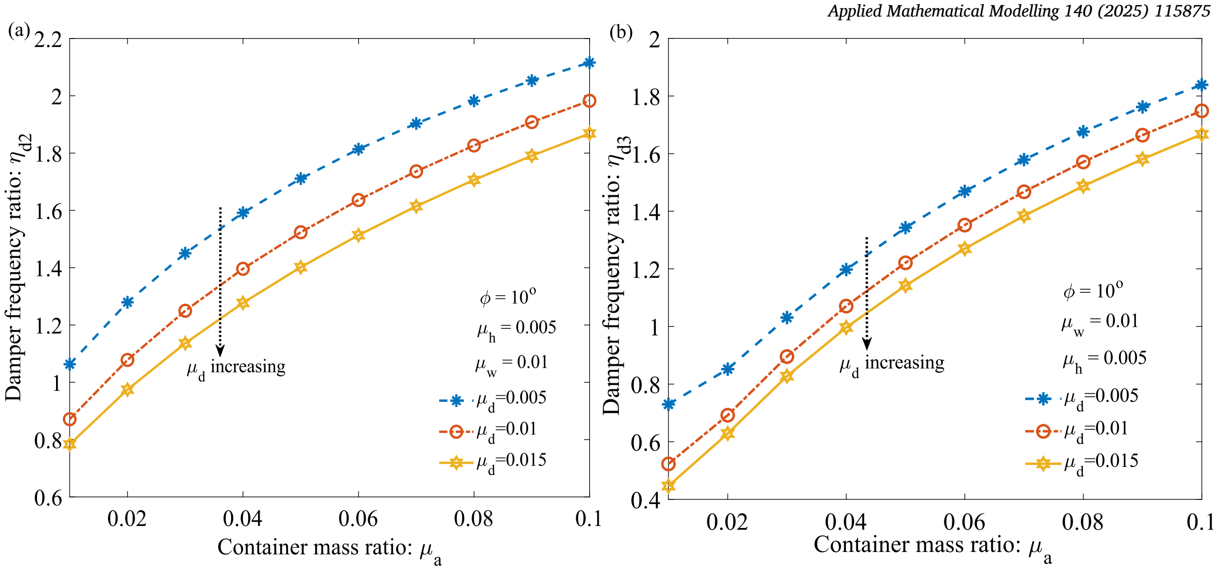

Fig. 5. The damper's frequency ratio as a function of container mass ratio. The amplifier's mass ratio is varied, i.e. $\mu_d = 0.005$, $\mu_d = 0.005$, and $\mu_d = 0.015$. Eq. (25) is applied in the Eq. (B.1), Eq. (B.2), Eq. (B.3), Eq. (B.4), Eq. (B.5), and the (a) second and (b) third expressions of Eq. (60) to obtain this graph. The amplifier's angle and mass ratios of the liquid inside the horizontal and total length of the container are taken constant, i.e. $\phi = 10^o$, $\mu_w = 0.01$, and $\mu_h = 0.005$.

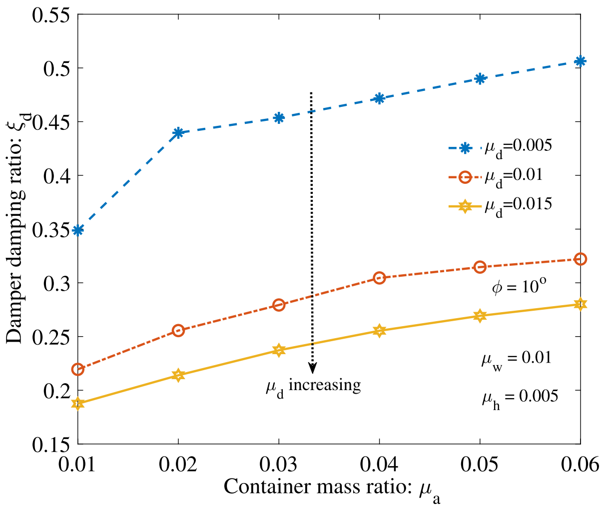

Fig. 6. The damping ratio of the H_∞ optimised damper is evaluated using Eq. (62) which is as a function of container mass ratio, i.e. μ_a. Equation (62) is applied to obtain the graph. The other system parameters are considered as $\mu_d = 0.005$, $\mu_d = 0.005$, and $\mu_d = 0.015$. $\phi = 10^o$, $\mu_w = 0.01$, and $\mu_h = 0.005$.

nonlinear inertial amplifier liquid column damper, an all-inclusive, moderate container mass ratio and a small amplifier mass ratio are needed.

4. Dynamic response reduction capacity evaluation

The newly developed nonlinear inertial amplifier liquid column dampers are applied to single-degree-of-freedom systems to attenuate their vibrations due to base excitation. H_2 and H_∞ optimised dampers are involved in this dynamic response reduction capacity evaluation process. The governing system parameters for all single-degree-of-freedom systems are considered the same. The values are listed in Table 1. In addition, the H_2 and H_∞ optimised design parameters are for this study are listed in Table 2 and Table 3. The optimal displacements of the single degree of freedom system controlled by the H_2 optimised liquid column dampers are obtained and graphically presented in Fig. 7 (a). An optimum conventional tuned liquid column damper and a tuned liquid column damper inerter are considered from a renowned published journal paper. The optimal system parameters for these conventional dampers are taken from this paper, cited in Table 2. Now, the maximum displacement response of each controlled structure is considered to evaluate the dynamic response reduction capacity of a nonlinear inertial amplifier liquid column damper with respect to the dynamic response

Table 1
Main dynamic system's structural design parameter.

Main dynamic system	Structural design parameter	Quantity
		ξ_s
Single-degree-of-freedom (SDOF)	Damping ratio	0.01

Table 2
The optimal design parameters of H_2 optimised liquid column dampers.

Damper	Proposed by	H_2 optimised design parameters			
		η_d	ξ_d	η_w	ξ_e
Nonlinear inertial amplifier liquid column damper	This study	0.754837	0.124082	1.04435	0.354333
Tuned liquid column damper inerter	Wang et al. [25]	2.39	0.4	1.04435	0.0588669
Conventional tuned liquid column damper	Wang et al. [25]	0.92	0	1.04435	0.0361704

The total mass ratio of each damper is considered the same to conduct a fair comparison among them in terms of dynamic response reduction capacity.

Table 3
H_∞ optimised design parameters for each optimum liquid column damper.

Damper	Proposed by	H_∞ optimised design parameters			
		η_d	ξ_d	η_w	ξ_e
Nonlinear inertial amplifier liquid column damper	This study	0.821583	0.489972	1.04435	0.161587
Tuned liquid column damper inerter	Matteo et al. [26]	1.8302	0.6029	0.4315	0.0150839
Conventional tuned liquid column damper	Zhao et al. [27]	0.98	0	1.05	0.04

The total mass ratio of each damper is considered the same to conduct a fair comparison among them in terms of dynamic response reduction capacity.

reduction capacity of the conventional tuned liquid column damper and tuned liquid column damper inerter. According to Fig. 7 (a), the maximum displacement of the uncontrolled single-degree-of-freedom system is obtained as 50. The maximum displacements of the single-degree-of-freedom systems controlled by the conventional tuned liquid column damper, tuned liquid column damper inerter, and nonlinear inertial amplifier liquid column damper are derived as 45.29, 20.53, and 10.41. A mathematical equation is derived to obtain the dynamic response reduction capacity and expressed below.

$$H_{dr}(\%) = \left(\frac{\left(H_s(\eta)\right)_{TLCD,TLCDI} - \left(H_s(\eta)\right)_{\text{NIALCD}}}{\left(H_s(\eta)\right)_{TLCD,TLCDI}} \right) \times 100 \tag{63}$$

In Eq. (63), $\left(H_s(\eta)\right)_{TLCD,TLCDI}$ defines the maximum displacements of single-degree-of-freedom systems controlled by the conventional tuned liquid column damper (TLCD) and tuned liquid column damper inerter (TLCDI). $\left(H_s(\eta)\right)_{\text{NIALCD}}$ defines the maximum displacement of the single-degree-of-freedom system controlled by the nonlinear inertial amplifier liquid column damper. The values are obtained: $\left(H_s(\eta)\right)_{\text{CTunedLiquidColumnDamper}} = 45.29$, $\left(H_s(\eta)\right)_{\text{TunedLiquidColumnDamperI}} = 20.53$, and $\left(H_s(\eta)\right)_{\text{NIALCD}} = 10.41$. Accordingly, the dynamic response reduction performance-wise, the H_2 optimised nonlinear inertial amplifier liquid column damper is 77.01% and 49.29% superior to the H_2 optimised conventional tuned liquid column damper and H_2 optimised tuned liquid column damper inerter. In a similar way, the H_∞ optimised damper's dynamic response reduction capacity has been produced. Hence, the graphical representation of the variations of the displacement responses of the single-degree-of-freedom systems controlled by H_∞ optimised dampers are shown in Fig. 7 (b). The optimal system parameters for the optimum conventional tuned liquid column dampers and tuned liquid column damper inerters are taken from two published journal papers, and those are cited in Table 3. The maximum displacement values are obtained: $\left(H_s(\eta)\right)_{\text{CTunedLiquidColumnDamper}} = 42.26$, $\left(H_s(\eta)\right)_{\text{TunedLiquidColumnDamperI}} = 38.74$, and $\left(H_s(\eta)\right)_{\text{NIALCD}} = 6.93$. Therefore, the H_∞ optimised nonlinear inertial amplifier liquid column damper has 83.60% and 82.11% more dynamic response reduction capacity than the H_∞ optimised conventional tuned liquid column damper and H_∞ optimised tuned liquid column damper inerter.

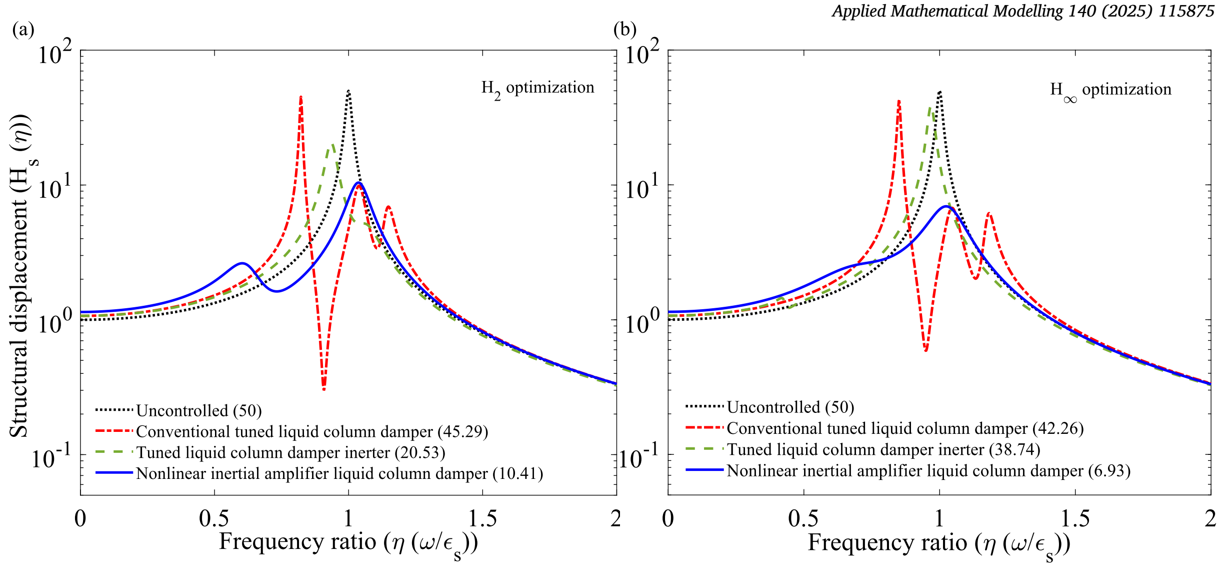

Fig. 7. The optimal displacements of the single degree of freedom system controlled by (a) H_2 and (b) H_∞ optimised liquid column dampers. The details of each optimal design parameter for both figures are listed in Table 1, Table 2, and Table 3.

5. Summary and conclusion

This work presents nonlinear inertial amplifier liquid column dampers as an innovative method to markedly improve vibration control in buildings without augmenting static mass. We formulated governing equations using Newton's second law and Lagrange's approach, then optimised the damper's design parameters employing H_2 and H_∞ optimisation strategies. The efficacy of the nonlinear inertial amplifier liquid column damper was shown by its application to a single degree of freedom (SDOF) system, wherein its performance was juxtaposed with that of conventional TLCDs and inerter-based TLCDs.

- The modified nonlinear inertial amplifier liquid column damper exhibited up to 83.60% enhanced vibration attenuation relative to conventional systems.
- The closed-form solutions were established, offering pragmatic design assistance for engineers.
- The suggested technology mitigates the bulk and flexibility constraints of conventional TLCDs without augmenting static mass.

This work's principal innovation is the introduction of Nonlinear Inertial Amplifier Liquid Column Dampers, which significantly improve upon conventional Tuned Liquid Column Dampers (TLCDs). Present TLCD designs need the incorporation of significant static mass to attain efficient vibration mitigation, which is often impracticable for many applications owing to heightened costs, structural flexibility concerns, and design constraints. Our suggested nonlinear inertial amplifier liquid column dampers architecture boosts effective mass by the use of nonlinear inertial amplifiers, without increasing physical static mass. This innovative method overcomes the shortcomings of conventional TLCDs and inerter-based TLCD systems, leading to markedly enhanced vibration attenuation.

We substantiate our research by demonstrating that, via H_2 and H_∞ optimisation, nonlinear inertial amplifier liquid column dampers attain a dynamic response reduction of up to 83.60% and 82.11% superior to conventional TLCDs and inerter-based TLCD configurations. This development has significant ramifications for civil engineering constructions, where effective, lightweight, and economical vibration control is essential. The closed-form solutions we provide for optimum design parameters validate that our technique is both theoretically robust and practically viable, constituting a significant addition to the area. These novel dampers eliminate the disadvantages of conventional tuned liquid column dampers and tuned liquid column damper inerters, which offer improved vibration mitigation without adding static mass and affordable anti-seismic solutions. The closed-form optimal designs that have been suggested can be implemented practically. These discoveries advance the field by offering closed-form solutions for optimum design parameters, making the nonlinear inertial amplifier liquid column dampers a practical and efficient option for civil engineering applications, such as buildings, bridges, and other major structures. The nonlinear inertial amplifier liquid column dampers design signifies a notable progression in vibration control, providing civil engineers with a more effective and pragmatic option for enhancing the robustness of tall buildings, bridges, and other structures. The capacity to get enhanced performance without augmenting static mass offers economic and technical advantages, possibly transforming the methodology of vibration control in large-scale structures. This study enhances the existing understanding of passive vibration control and paves the way for future research opportunities. The nonlinear inertial amplifier liquid column dampers idea may be applied to multi-degree-of-freedom systems, and its efficacy under various dynamic loading circumstances warrants further study. This study provides a lightweight, economical, and high-performance damping solution, facilitating the development of more robust and sustainable structural systems.

CRediT authorship contribution statement

Sudip Chowdhury: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Sondipon Adhikari:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The closed-form expression from Eq. (19), Eq. (21), Eq. (22), and Eq. (28)

The closed-form expression for N_1 from Eq. (19) is listed below.

$$N_1 = \left(\begin{array}{c}
4\,\mu_{cd0}{}^5\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 - 8\,\mu_{cd0}{}^5\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
-8\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 8\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
+4\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + \mu_{cd0}{}^4\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+\mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 2\,\mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + \mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
-2\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 + 2\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
-24\,\mu_{cd0}{}^4\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 12\,\mu_{cd0}{}^4\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
-24\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 24\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
+12\,\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^4\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 12\,\mu_{cd0}{}^4\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
+4\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 8\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + 4\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
+\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 - 2\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + \mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
+\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^6 - 10\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+9\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - 2\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
+2\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 12\,\mu_{cd0}{}^3\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-24\,\mu_{cd0}{}^3\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 12\,\mu_{cd0}{}^3\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
-2\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^6 + 6\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 - 12\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
+16\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + \mu_{cd0}{}^2\mu_{dh0}{}^6\epsilon_d{}^4\epsilon_s{}^6 \\
+6\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 12\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
+6\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 - 12\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+12\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + 4\,\mu_{cd0}{}^2\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-8\,\mu_{cd0}{}^2\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 4\,\mu_{cd0}{}^2\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
+6\,\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 + 4\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
-8\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + 4\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
-3\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 6\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
-3\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 - 4\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+5\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + 6\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-6\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 - 3\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
+\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
+\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 - 2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 4\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
-2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 + \mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^8\epsilon_w{}^2 - 2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^6\epsilon_w{}^4 \\
+\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^4\epsilon_w{}^6 + 2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_s{}^8\epsilon_w{}^2 + 2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_s{}^6\epsilon_w{}^4 + \mu_{de0}\,\mu_{dh0}{}^4\epsilon_s{}^8\epsilon_w{}^2 \\
+4\,\mu_{cd0}{}^5\mu_{de0}{}^3\xi_d{}^2\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 - 2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4
\end{array} \right). \tag{A.1}$$

The closed-form expression for N_2 and N_3 from Eq. (21) is listed below.

$$N_2 = \left(\begin{array}{c}
-\mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 + 2\,\mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - \mu_{cd0}{}^6\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
+2\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 2\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
-\mu_{cd0}{}^4\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 6\,\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
-4\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 + 8\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - 4\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
-\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 2\,\mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 - \mu_{cd0}{}^5\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
-\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^6 + 10\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
-9\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 + 2\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-2\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 2\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^6 \\
-6\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - \mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-\mu_{cd0}{}^2\mu_{dh0}{}^6\epsilon_d{}^4\epsilon_s{}^6 - 6\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+12\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - 6\,\mu_{cd0}{}^4\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
+12\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 12\,\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
-4\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 + 8\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - 4\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 \\
+3\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 - 6\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 3\,\mu_{cd0}{}^3\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 \\
+4\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 - 5\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 \\
-6\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 + 6\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 \\
+3\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^4\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 - \mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^4\epsilon_w{}^2 \\
+2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_s{}^2\epsilon_w{}^4 - \mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^4\epsilon_w{}^6 + 2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
-4\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_d{}^2\epsilon_s{}^2\epsilon_w{}^6 - \mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^8\epsilon_w{}^2 \\
+2\,\mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^6\epsilon_w{}^4 - \mu_{cd0}{}^2\mu_{de0}{}^3\epsilon_s{}^4\epsilon_w{}^6 - 2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^6\epsilon_w{}^2 \\
+2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_d{}^2\epsilon_s{}^4\epsilon_w{}^4 + 2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_s{}^8\epsilon_w{}^2 \\
-2\,\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^2\epsilon_s{}^6\epsilon_w{}^4 - \mu_{de0}\,\mu_{dh0}{}^4\epsilon_s{}^8\epsilon_w{}^2
\end{array}\right). \tag{A.2}$$

$$N_3 = \frac{-4\,\epsilon_s{}^2\left(\begin{array}{c}
\left(\epsilon_s-\epsilon_w\right)^2\left(\epsilon_s+\epsilon_w\right)^2\left(\mu_{cd0}+1\right)^3\mu_{de0}{}^2 \\
-2\,\epsilon_s{}^2\mu_{dh0}{}^2\left(\begin{array}{c}\left(\mu_{cd0}{}^2+3\,\mu_{cd0}+3/2\right)\epsilon_s{}^2 \\ -\epsilon_w{}^2\left(\mu_{cd0}+2\right)\left(\mu_{cd0}+1\right)\end{array}\right)\mu_{de0} \\
+\mu_{dh0}{}^4\epsilon_s{}^4\left(\mu_{cd0}+3\right)
\end{array}\right)}{\mu_{de0}\,\mu_{cd0}{}^2\epsilon_w{}^2\epsilon_d{}^2}. \tag{A.3}$$

The closed-form expression for N_4 from Eq. (22) is listed below.

$$N_4 = \mu_{de0}\,\epsilon_d\,\mu_{cd0}\,\epsilon_w{}^2\left(\mu_{cd0}\,\mu_{de0}\,\epsilon_s{}^2 - \mu_{cd0}\,\mu_{de0}\,\epsilon_w{}^2 - \mu_{dh0}{}^2\epsilon_s{}^2\right)^2\epsilon_s{}^6\sqrt{\frac{N_2}{N_3}}\,. \tag{A.4}$$

The closed-form expression for N_5, N_6, and N_7 from Eq. (28) is listed below.

$$N_5 = \left(\begin{array}{c}
343\,\mu_{cd0}{}^{15}\mu_{de0}{}^3 - 1029\,\mu_{cd0}{}^{14}\mu_{de0}{}^2\mu_{dh0}{}^2 + 1029\,\mu_{cd0}{}^{13}\mu_{de0}\,\mu_{dh0}{}^4 \\
-343\,\mu_{cd0}{}^{12}\mu_{dh0}{}^6 - 2254\,\mu_{cd0}{}^{14}\mu_{de0}{}^3 + 6762\,\mu_{cd0}{}^{13}\mu_{de0}{}^2\mu_{dh0}{}^2 \\
-6762\,\mu_{cd0}{}^{12}\mu_{de0}\,\mu_{dh0}{}^4 + 2254\,\mu_{cd0}{}^{11}\mu_{dh0}{}^6 - 896\,\mu_{cd0}{}^{13}\mu_{de0}{}^3 \\
+2884\,\mu_{cd0}{}^{12}\mu_{de0}{}^2\mu_{dh0}{}^2 - 3080\,\mu_{cd0}{}^{11}\mu_{de0}\,\mu_{dh0}{}^4 + 1092\,\mu_{cd0}{}^{10}\mu_{dh0}{}^6 \\
+18646\,\mu_{cd0}{}^{12}\mu_{de0}{}^3 - 56890\,\mu_{cd0}{}^{11}\mu_{de0}{}^2\mu_{dh0}{}^2 \\
+57842\,\mu_{cd0}{}^{10}\mu_{de0}\,\mu_{dh0}{}^4 - 19598\,\mu_{cd0}{}^9\mu_{dh0}{}^6 \\
+18725\,\mu_{cd0}{}^{11}\mu_{de0}{}^3 - 58297\,\mu_{cd0}{}^{10}\mu_{de0}{}^2\mu_{dh0}{}^2 \\
+60454\,\mu_{cd0}{}^9\mu_{de0}\,\mu_{dh0}{}^4 - 20882\,\mu_{cd0}{}^8\mu_{dh0}{}^6 - 47296\,\mu_{cd0}{}^{10}\mu_{de0}{}^3 \\
+152231\,\mu_{cd0}{}^9\mu_{de0}{}^2\mu_{dh0}{}^2 - 162832\,\mu_{cd0}{}^8\mu_{de0}\,\mu_{dh0}{}^4 \\
+57897\,\mu_{cd0}{}^7\mu_{dh0}{}^6 - 128348\,\mu_{cd0}{}^9\mu_{de0}{}^3 \\
+395804\,\mu_{cd0}{}^8\mu_{de0}{}^2\mu_{dh0}{}^2 - 406383\,\mu_{cd0}{}^7\mu_{de0}\,\mu_{dh0}{}^4 \\
+138929\,\mu_{cd0}{}^6\mu_{dh0}{}^6 - 143960\,\mu_{cd0}{}^8\mu_{de0}{}^3 + 411491\,\mu_{cd0}{}^7\mu_{de0}{}^2\mu_{dh0}{}^2 \\
-389340\,\mu_{cd0}{}^6\mu_{de0}\,\mu_{dh0}{}^4 + 121785\,\mu_{cd0}{}^5\mu_{dh0}{}^6 - 97024\,\mu_{cd0}{}^7\mu_{de0}{}^3 \\
+258496\,\mu_{cd0}{}^6\mu_{de0}{}^2\mu_{dh0}{}^2 - 227480\,\mu_{cd0}{}^5\mu_{de0}\,\mu_{dh0}{}^4 \\
+66104\,\mu_{cd0}{}^4\mu_{dh0}{}^6 - 42304\,\mu_{cd0}{}^6\mu_{de0}{}^3 + 128\,\mu_{dh0}{}^6 \\
+109368\,\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^2 - 95056\,\mu_{cd0}{}^4\mu_{de0}\,\mu_{dh0}{}^4 \\
+27864\,\mu_{cd0}{}^3\mu_{dh0}{}^6 - 11696\,\mu_{cd0}{}^5\mu_{de0}{}^3 + 29904\,\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^2 \\
-25104\,\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^4 + 6896\,\mu_{cd0}{}^2\mu_{dh0}{}^6 - 1856\,\mu_{cd0}{}^4\mu_{de0}{}^3 \\
+5184\,\mu_{dh0}{}^2\mu_{de0}{}^2\mu_{cd0}{}^3 - 4800\,\mu_{dh0}{}^4\mu_{de0}\,\mu_{cd0}{}^2 + 1472\,\mu_{dh0}{}^6\mu_{cd0} \\
-128\,\mu_{cd0}{}^3\mu_{de0}{}^3 + 384\,\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^2 - 384\,\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^4
\end{array}\right). \tag{A.5}$$

$$N_6 = \begin{pmatrix} 7\,\mu_{cd0}{}^6\mu_{de0} - 7\,\mu_{cd0}{}^5\mu_{dh0}{}^2 - 29\,\mu_{cd0}{}^5\mu_{de0} + 29\,\mu_{cd0}{}^4\mu_{dh0}{}^2 \\ -12\,\mu_{cd0}{}^4\mu_{de0} + 13\,\mu_{cd0}{}^3\mu_{dh0}{}^2 + 60\,\mu_{cd0}{}^3\mu_{de0} - 73\,\mu_{cd0}{}^2\mu_{dh0}{}^2 \\ +44\,\mu_{cd0}{}^2\mu_{de0} - 8\,\mu_{cd0}\,\mu_{dh0}{}^2 + 8\,\mu_{cd0}\,\mu_{de0} - 8\,\mu_{dh0}{}^2 \end{pmatrix}. \tag{A.6}$$

$$N_7 = \begin{pmatrix} 7\,\mu_{cd0}{}^5\mu_{de0} - 7\,\mu_{cd0}{}^4\mu_{dh0}{}^2 - 16\,\mu_{cd0}{}^4\mu_{de0} + 16\,\mu_{cd0}{}^3\mu_{dh0}{}^2 \\ -35\,\mu_{cd0}{}^3\mu_{de0} + 37\,\mu_{cd0}{}^2\mu_{dh0}{}^2 - 20\,\mu_{cd0}{}^2\mu_{de0} + 12\,\mu_{cd0}\,\mu_{dh0}{}^2 \\ -8\,\mu_{cd0}\,\mu_{de0} + 8\,\mu_{dh0}{}^2 \end{pmatrix}. \tag{A.7}$$

Appendix B. The closed-form expressions from Eq. (59) and Eq. (62)

The closed-form expressions from Eq. (59) are listed below.

$$W_1 = \frac{8\left(\left(\mu_{cd0}+1\right)\left(\eta_w{}^2\mu_{cd0}+\eta_w{}^2+1\right)\mu_{de0} - 1/2\,\mu_{dh0}{}^2\right)}{\left(\left(\mu_{cd0}+1\right)\mu_{de0}-\mu_{dh0}{}^2\right)^3\left(\eta_w{}^2\mu_{cd0}+\eta_w{}^2+1\right)\mu_{cd0}{}^2}. \tag{B.1}$$

$$W_2 = \frac{16\left(\left(\mu_{cd0}+1\right)\mu_{de0}-\mu_{dh0}{}^2\right)\mu_{cd0}}{\left(\eta_w{}^2\mu_{cd0}+\eta_w{}^2+1\right)\left(\mu_{cd0}+1\right)} \begin{pmatrix} \begin{pmatrix} \left(\eta_w{}^4+3/4\,\eta_w{}^2\right)\mu_{cd0}{}^2 \\ +\left(2\,\eta_w{}^4+9/4\,\eta_w{}^2+1/4\right)\mu_{cd0} \\ +\eta_w{}^4+3/2\,\eta_w{}^2+1/2 \end{pmatrix}\mu_{cd0}\,\mu_{de0}{}^3 \\ -\begin{pmatrix} \left(\eta_w{}^6+3/2\,\eta_w{}^4\right)\mu_{cd0}{}^4 \\ +\left(\frac{13\,\eta_w{}^6}{4}+\frac{27\,\eta_w{}^4}{4}+9/4\,\eta_w{}^2\right)\mu_{cd0}{}^3 \\ +\left(\frac{15\,\eta_w{}^6}{4}+19/2\,\eta_w{}^4+\frac{25\,\eta_w{}^2}{4}+5/8\right)\mu_{cd0}{}^2 \\ +\left(7/4\,\eta_w{}^6+\frac{19\,\eta_w{}^4}{4}+\frac{19\,\eta_w{}^2}{4}+5/4\right)\mu_{cd0} \\ +1/4\,\left(\eta_w{}^2+1\right)\left(\eta_w{}^4+\eta_w{}^2+2\right) \end{pmatrix}\mu_{dh0}{}^2\mu_{de0}{}^2 \\ +3/4\begin{pmatrix} \eta_w{}^4\mu_{cd0}{}^3+\left(8/3\,\eta_w{}^4+2\,\eta_w{}^2\right)\mu_{cd0}{}^2 \\ +\left(5/3\,\eta_w{}^4+3\,\eta_w{}^2+2/3\right)\mu_{cd0} \\ +1/2\,\eta_w{}^2+2/3 \end{pmatrix}\mu_{dh0}{}^4\mu_{de0} \\ -1/4\,\left(\eta_w{}^2\mu_{cd0}+1/2\right)\mu_{dh0}{}^6 \end{pmatrix}. \tag{B.2}$$

$$W_3 = \begin{matrix} 8\,\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 - 8\,\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 32\,\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -28\,\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 48\,\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^4 - 36\,\eta_w{}^8\mu_{cd0}{}^3\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +16\,\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 - 32\,\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 16\,\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 \\ +32\,\eta_w{}^8\mu_{cd0}{}^3\mu_{de0}{}^4 - 20\,\eta_w{}^8\mu_{cd0}{}^2\mu_{de0}{}^3\mu_{dh0}{}^2 + 80\,\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -124\,\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 44\,\eta_w{}^6\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^4 + 8\,\eta_w{}^8\mu_{cd0}{}^2\mu_{de0}{}^4 \\ -4\,\eta_w{}^8\mu_{cd0}\,\mu_{de0}{}^3\mu_{dh0}{}^2 + 144\,\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^4 - 164\,\eta_w{}^6\mu_{cd0}{}^3\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +34\,\eta_w{}^6\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^4 + 6\,\eta_w{}^4\mu_{cd0}{}^6\mu_{de0}{}^4 - 18\,\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +18\,\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 - 6\,\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}\,\mu_{dh0}{}^6 + 112\,\eta_w{}^6\mu_{cd0}{}^3\mu_{de0}{}^4 \\ -84\,\eta_w{}^6\mu_{cd0}{}^2\mu_{de0}{}^3\mu_{dh0}{}^2 + 6\,\eta_w{}^6\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^4 + 44\,\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -104\,\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 76\,\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^4 - 16\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^6 \\ +32\,\eta_w{}^6\mu_{cd0}{}^2\mu_{de0}{}^4 - 12\,\eta_w{}^6\mu_{cd0}\,\mu_{de0}{}^3\mu_{dh0}{}^2 + 110\,\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^4 \\ -196\,\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}{}^3\mu_{dh0}{}^2 + 94\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^4 - 8\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^6 \\ +112\,\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}{}^4 - 138\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^3\mu_{dh0}{}^2 + 36\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^4 \\ -2\,\eta_w{}^4\mu_{de0}\,\mu_{dh0}{}^6 + 4\,\eta_w{}^2\mu_{cd0}{}^5\mu_{de0}{}^4 - 13\,\eta_w{}^2\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +15\,\eta_w{}^2\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^4 - 7\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}\,\mu_{dh0}{}^6 + \eta_w{}^2\mu_{cd0}\,\mu_{dh0}{}^8 \\ +40\,\eta_w{}^4\mu_{cd0}{}^2\mu_{de0}{}^4 - 28\,\eta_w{}^4\mu_{cd0}\,\mu_{de0}{}^3\mu_{dh0}{}^2 + 4\,\eta_w{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 \\ +20\,\eta_w{}^2\mu_{cd0}{}^4\mu_{de0}{}^4 - 52\,\eta_w{}^2\mu_{cd0}{}^3\mu_{de0}{}^3\mu_{dh0}{}^2 + 45\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^2\mu_{dh0}{}^4 \\ -14\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}{}^6 + \eta_w{}^2\mu_{dh0}{}^8 + 32\,\eta_w{}^2\mu_{cd0}{}^3\mu_{de0}{}^4 \\ -62\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^3\mu_{dh0}{}^2 + 34\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^2\mu_{dh0}{}^4 - 4\,\eta_w{}^2\mu_{de0}\,\mu_{dh0}{}^6 \\ +16\,\eta_w{}^2\mu_{cd0}{}^2\mu_{de0}{}^4 - 20\,\eta_w{}^2\mu_{cd0}\,\mu_{de0}{}^3\mu_{dh0}{}^2 + 4\,\eta_w{}^2\mu_{de0}{}^2\mu_{dh0}{}^4 \end{matrix}. \tag{B.3}$$

$$W_4 = \frac{4\begin{pmatrix} {\mu_{cd0}}^2\left(\left({\eta_w}^2+1/2\right)\mu_{cd0}+{\eta_w}^2+1\right){\mu_{de0}}^2 \\ -{\mu_{dh0}}^2\mu_{cd0}\left(\left({\eta_w}^2+1\right)\mu_{cd0}+1/2\,{\eta_w}^2+3/2\right)\mu_{de0} \\ +1/2\,{\mu_{dh0}}^4\left(\mu_{cd0}+1\right)\end{pmatrix}}{\mu_{de0}\,{\eta_w}^4\left(\left(\mu_{cd0}\,{\eta_w}^2+\mu_{cd0}/2+{\eta_w}^2+1\right)\mu_{de0}-1/2\,{\mu_{dh0}}^2\right)\left(\mu_{cd0}+2\right)}. \tag{B.4}$$

The closed-form expression for the W_5 is derived as

$$W_5 = \frac{12\sqrt{3}\sqrt{\begin{matrix}27\,{W_1}^2{W_4}^2+\left(-18\,W_3\,W_2\,W_1+4\,{W_2}^3\right)W_4 \\ +4\,W_1\,{W_3}^3-{W_2}^2{W_3}^2\end{matrix}}\,W_1}{-108\,W_4\,{W_1}^2+36\,W_3\,W_2\,W_1-8\,{W_2}^3}. \tag{B.5}$$

The closed-form expressions for Z_1, Z_2, and Z_3 from Eq. (62) are listed below.

$$Z_1 = \frac{-32\,{\eta_d}^4\left(\left(\mu_{cd0}+1\right)\left(-{\eta_w}^2+\eta_{1,2,3,4}^2\right)\mu_{de0}-\eta_{1,2,3,4}^2\,{\mu_{dh0}}^2\right)\begin{pmatrix}\left(\mu_{cd0}+1\right)^2\left(-{\eta_w}^2+\eta_{1,2,3,4}^2\right)^2{\mu_{de0}}^2+\eta_{1,2,3,4}^4{\mu_{dh0}}^4 \\ -2\left(\left(\mu_{cd0}+1\right)\eta_{1,2,3,4}^4-{\eta_w}^2\left(\mu_{cd0}+1\right)\eta_{1,2,3,4}^2-1/2\,{\eta_w}^2\right){\mu_{dh0}}^2\mu_{de0}\end{pmatrix}}{\eta_{1,2,3,4}^4\left(\left(-1+\left(\mu_{cd0}+1\right)\eta_{1,2,3,4}^2\right)\left(-{\eta_w}^2+\eta_{1,2,3,4}^2\right)\mu_{de0}-\eta_{1,2,3,4}^4{\mu_{dh0}}^2\right){\mu_{cd0}}^4}. \tag{B.6}$$

$$Z_2 = \frac{\begin{matrix}-16\,{\eta_d}^2{\mu_{cd0}}^2\left(\left(\mu_{cd0}+1\right)\mu_{de0}-{\mu_{dh0}}^2\right)^2\left(\mu_{cd0}\,\mu_{de0}-{\mu_{dh0}}^2\right)^2\eta_{1,2,3,4}^{16} \\ +\left(32\,{\eta_d}^2\begin{pmatrix}\left(\mu_{cd0}+1\right) \\ \left({\eta_d}^2\mu_{cd0}+{\eta_d}^2+2\,{\eta_w}^2+1/2\right)\mu_{cd0}\,{\mu_{de0}}^2 \\ -2\,{\mu_{dh0}}^2\begin{pmatrix}{\eta_d}^2{\mu_{cd0}}^2 \\ +\left({\eta_d}^2+{\eta_w}^2+1/2\right)\mu_{cd0} \\ +1/2\,{\eta_w}^2+1/4\end{pmatrix}\mu_{de0} \\ +{\mu_{dh0}}^4\left({\eta_d}^2\mu_{cd0}+1/2\right)\end{pmatrix}\right)\eta_{1,2,3,4}^{14} \\ +P_1\eta_{1,2,3,4}^{12}+P_2\eta_{1,2,3,4}^{10}+P_3\eta_{1,2,3,4}^{8}+P_4\eta_{1,2,3,4}^{6}+P_5\eta_{1,2,3,4}^{4} \\ +16\,\left(\mu_{cd0}+1\right){\eta_d}^6{\mu_{de0}}^3{\eta_w}^6\left({\eta_w}^2\left(\mu_{cd0}+1\right)^2\mu_{de0}+{\mu_{dh0}}^2\right){\mu_{cd0}}^4\eta_{1,2,3,4}^2\end{matrix}}{\left(\left(\mu_{cd0}+1\right)\mu_{de0}-{\mu_{dh0}}^2\right)\left(\mu_{cd0}\,\mu_{de0}-{\mu_{dh0}}^2\right){\mu_{cd0}}^2}. \tag{B.7}$$

The closed-form expressions for P_1, P_2, P_3, P_4, and P_5 are listed below.

$$Z_3 = \frac{\begin{matrix}-2\,\left(\mu_{cd0}\,\mu_{de0}-{\mu_{dh0}}^2\right)^4\eta_{1,2,3,4}^{18} \\ +8\,\left(\mu_{cd0}\,\mu_{de0}-{\mu_{dh0}}^2\right)^3 \\ \begin{pmatrix}{\eta_d}^2{\mu_{cd0}}^2\mu_{de0}+\left(\left({\eta_d}^2+{\eta_w}^2+1/4\right)\mu_{de0}-{\eta_d}^2{\mu_{dh0}}^2\right)\mu_{cd0} \\ -1/4\,{\mu_{dh0}}^2\end{pmatrix}\eta_{1,2,3,4}^{16} \\ -\left(\begin{matrix}-12\,\left(\mu_{cd0}\,\mu_{de0}-{\mu_{dh0}}^2\right)^2\mu_{cd0} \\ \left(\begin{matrix}\mu_{cd0}\begin{pmatrix}{\eta_d}^4{\mu_{cd0}}^2 \\ +2\,{\eta_d}^2\left({\eta_d}^2+4/3\,{\eta_w}^2+1/3\right)\mu_{cd0} \\ +{\eta_d}^4+\left(8/3\,{\eta_w}^2+2/3\right){\eta_d}^2 \\ +{\eta_w}^4+2/3\,{\eta_w}^2\end{pmatrix}{\mu_{de0}}^2 \\ -2\begin{pmatrix}{\eta_d}^4{\mu_{cd0}}^2 \\ +{\eta_d}^2\left({\eta_d}^2+4/3\,{\eta_w}^2+2/3\right)\mu_{cd0} \\ +\left(1/3\,{\eta_w}^2+1/3\right){\eta_d}^2+1/3\,{\eta_w}^2\end{pmatrix}{\mu_{dh0}}^2\mu_{de0} \\ +{\eta_d}^2{\mu_{dh0}}^4\left({\eta_d}^2\mu_{cd0}+2/3\right)\end{matrix}\right)\end{matrix}\right)\eta_{1,2,3,4}^{14} \\ +P_6\eta_{1,2,3,4}^{12}+P_7\eta_{1,2,3,4}^{10}+P_8\eta_{1,2,3,4}^{8}+P_9\eta_{1,2,3,4}^{6}+P_{10}\eta_{1,2,3,4}^{4}+P_{11}\eta_{1,2,3,4}^{2}+ \\ 2\,\left(\mu_{cd0}+1\right){\mu_{de0}}^3{\eta_d}^6{\mu_{cd0}}^4{\eta_w}^6\end{matrix}}{\left({\eta_d}^2{\eta_w}^2{\mu_{cd0}}^2\mu_{de0}+2\,\mu_{de0}\,\left({\eta_d}^2+1/2\right){\eta_w}^2\mu_{cd0}+{\eta_d}^2\left({\eta_w}^2\mu_{de0}+{\mu_{dh0}}^2\right)\right)}. \tag{B.8}$$

The closed-form expressions for P_6, P_7, P_8, P_9, P_{10}, and P_{11} are listed below. The closed-form expressions for P_1, P_2, P_3, P_4, and P_5 from Eq. (B.7) are listed below.

$$P_1 = \begin{array}{c}
-16\,\eta_d^{6}\mu_{cd0}^{8}\mu_{de0}^{4}+64\,\eta_d^{6}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2}-96\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-16\,\eta_d^{6}\mu_{cd0}^{4}\mu_{dh0}^{8}-64\,\eta_d^{6}\mu_{cd0}^{7}\mu_{de0}^{4}+192\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-192\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}+64\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}-128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4}\\
+384\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-384\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}+128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}\\
-96\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{4}+192\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-96\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-384\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}+864\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-576\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}+192\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-64\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{4}+64\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-384\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}+576\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-192\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-8\,\eta_d^{4}\mu_{cd0}^{7}\mu_{de0}^{4}+32\,\eta_d^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-48\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+32\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}-8\,\eta_d^{4}\mu_{cd0}^{3}\mu_{dh0}^{8}-192\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+288\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}-16\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{4}\\
-128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4}+96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}-48\,\eta_d^{4}\mu_{cd0}^{6}\mu_{de0}^{4}\\
+144\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-144\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+48\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}\\
-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}+96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}-16\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-64\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}+192\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-192\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+64\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}-72\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{4}+144\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-72\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}-128\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}+296\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-208\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+40\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6}-32\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{4}\\
+32\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}-64\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4}+96\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-32\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4}-4\,\eta_d^{2}\mu_{cd0}^{6}\mu_{de0}^{4}+16\,\eta_d^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-24\,\eta_d^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+16\,\eta_d^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}-4\,\eta_d^{2}\mu_{cd0}^{2}\mu_{dh0}^{8}\\
-8\,\eta_d^{2}\mu_{cd0}^{5}\mu_{de0}^{4}+24\,\eta_d^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-24\,\eta_d^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+8\,\eta_d^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6}+64\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}\,\mu_{dh0}^{6}
\end{array}. \tag{B.9}$$

$$P_2 = \begin{array}{c}
64\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{8}\mu_{de0}^{4}-192\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2}+192\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-64\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}\,\mu_{dh0}^{6}+256\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4}-576\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+384\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}-64\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}+192\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4}\\
-384\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}+192\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}+384\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-576\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+192\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+576\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-864\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+288\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+64\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-64\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+256\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}-192\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+16\,\eta_d^{6}\mu_{cd0}^{7}\mu_{de0}^{4}-48\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}+48\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-16\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}+576\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}-576\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+96\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+32\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4}-96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}-32\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6}+128\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}\\
-96\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}+64\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4}+48\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-96\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+48\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+192\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}\\
-96\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}+192\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}-416\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-32\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}+64\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}-32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}-192\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+48\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{4}-48\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}+288\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+192\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}-304\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+112\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+16\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{4}+128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4}\\
-96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}+96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+16\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4}+16\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}-48\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+48\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-16\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6}+32\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}\\
-80\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}+64\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}-16\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6}\\
+256\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-416\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}
\end{array}. \tag{B.10}$$

$$
P_3 = \begin{array}{c}
-96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{8}\mu_{de0}^{4}+192\eta_d^{6}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2}-96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-384\eta_d^{6}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4}+576\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-192\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{4}+128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-576\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}\\
+576\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-384\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-16\eta_d^{2}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}-384\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}+192\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-64\eta_d^{6}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4}+128\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-64\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-384\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}+192\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-48\eta_d^{4}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4}\\
+96\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}-48\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}-32\eta_d^{2}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}\\
-96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}-192\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4}+256\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-64\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}+32\eta_d^{4}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-288\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}+400\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-112\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-16\eta_d^{2}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4}-64\eta_d^{2}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}+64\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-192\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4}+128\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-432\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+400\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-56\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}-128\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+104\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-64\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4}-192\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}\\
+96\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}-8\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-8\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\mu_{dh0}^{6}-64\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}+32\eta_d^{2}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-24\eta_d^{2}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}+48\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-24\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}\\
-16\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}-48\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+88\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-40\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+288\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}
\end{array}. \tag{B.11}
$$

$$
P_4 = \begin{array}{c}
64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{8}\mu_{de0}^{4}-64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2}+256\eta_d^{6}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{4}\\
+32\eta_d^{4}\eta_w^{8}\mu_{cd0}^{7}\mu_{de0}^{4}+384\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}-192\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+96\eta_d^{4}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}+256\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}-64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4}-112\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4}\\
+96\eta_d^{4}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}+32\eta_d^{4}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{4}-32\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}+288\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4}-224\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+16\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+32\eta_d^{4}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4}+192\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{2}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}+288\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4}\\
-112\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}+288\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}-128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+32\eta_d^{2}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}+96\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4}+16\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-16\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+128\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}-32\eta_d^{4}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+16\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-16\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}+16\eta_d^{2}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4}\\
+16\eta_d^{2}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}-16\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
+32\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-16\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}+32\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}\\
-32\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-192\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}
\end{array}. \tag{B.12}
$$

$$
P_5 = \begin{array}{c}
-16\eta_d^{6}\eta_w^{8}\mu_{cd0}^{8}\mu_{de0}^{4}-64\eta_d^{6}\eta_w^{8}\mu_{cd0}^{7}\mu_{de0}^{4}-96\eta_d^{6}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}\\
-64\eta_d^{6}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}-64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{4}+32\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-8\eta_d^{4}\eta_w^{8}\mu_{cd0}^{7}\mu_{de0}^{4}-16\eta_d^{6}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4}-192\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4}\\
+64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-48\eta_d^{4}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}-192\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4}\\
+32\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-72\eta_d^{4}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}-64\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4}\\
-32\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}+16\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4}-32\eta_d^{4}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4}\\
-8\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2}-4\eta_d^{2}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4}-32\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}\\
-16\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}-8\eta_d^{2}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4}
\end{array}. \tag{B.13}
$$

The closed-form expressions for P_6, P_7, P_8, P_9, P_{10}, and P_{11} from Eq. (B.8) are listed below.

$$P_6 = \begin{array}{c}
8\,\eta_d^{6}\mu_{cd0}^{7}\mu_{de0}^{4} - 32\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} + 48\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-32\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} + 8\,\eta_d^{6}\mu_{cd0}^{3}\mu_{dh0}^{8} + 24\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-72\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 72\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 24\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} \\
+48\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} - 144\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 144\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-48\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} + 24\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{4} - 48\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+24\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 96\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} - 216\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+144\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 24\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} + 48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 8\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-8\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 48\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} - 72\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+24\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} + 10\,\eta_d^{4}\mu_{cd0}^{6}\mu_{de0}^{4} - 40\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+60\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 40\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} + 10\,\eta_d^{4}\mu_{cd0}^{2}\mu_{dh0}^{8} \\
+48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} - 72\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 24\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+8\,\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} - 8\,\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 22\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-66\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 66\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 22\,\eta_d^{4}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} \\
+32\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} - 96\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 96\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-32\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} + 12\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{4} - 24\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+12\,\eta_d^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} + 32\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} - 72\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+48\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{2}\eta_w^{2}\mu_{cd0}\,\mu_{de0}\,\mu_{dh0}^{6} + 12\,\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-24\,\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 12\,\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} + 2\,\eta_d^{2}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-8\,\eta_d^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 12\,\eta_d^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} \\
+2\,\eta_d^{2}\mu_{cd0}\,\mu_{dh0}^{8}
\end{array}. \tag{B.14}$$

$$P_7 = \begin{array}{c}
-2\,\eta_d^{8}\mu_{cd0}^{8}\mu_{de0}^{4} + 8\,\eta_d^{8}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{8}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+8\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}\,\mu_{dh0}^{6} - 2\,\eta_d^{8}\mu_{cd0}^{4}\mu_{dh0}^{8} - 8\,\eta_d^{8}\mu_{cd0}^{7}\mu_{de0}^{4} \\
+24\,\eta_d^{8}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} + 8\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} \\
-32\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4} + 96\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} - 96\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+32\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} - 12\,\eta_d^{8}\mu_{cd0}^{6}\mu_{de0}^{4} + 24\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-12\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 96\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} + 216\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-144\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 24\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} - 72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+144\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+8\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 96\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} + 144\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-48\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 4\,\eta_d^{6}\mu_{cd0}^{7}\mu_{de0}^{4} + 16\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-24\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} + 16\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} - 4\,\eta_d^{6}\mu_{cd0}^{3}\mu_{dh0}^{8} \\
-144\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} + 216\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} + 32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 2\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-32\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} + 24\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 16\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+48\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 48\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 16\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} \\
-72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} + 72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-40\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} + 120\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 120\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+40\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} - 32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} + 24\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-2\,\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4} - 20\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{4} + 40\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-20\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 88\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} + 196\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-128\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 20\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} - 48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+96\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{4} \\
+8\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 48\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} + 72\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-24\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} - 2\,\eta_d^{4}\mu_{cd0}^{6}\mu_{de0}^{4} + 8\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-12\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 8\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} - 2\,\eta_d^{4}\mu_{cd0}^{2}\mu_{dh0}^{8} \\
-48\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} + 72\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-8\,\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} + 8\,\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 4\,\eta_d^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+12\,\eta_d^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 4\,\eta_d^{4}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} \\
-8\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} + 24\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+8\,\eta_d^{2}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6}
\end{array}. \tag{B.15}$$

$$
P_8 = \begin{array}{c}
8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{8}\mu_{de0}^{4} - 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2} + 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}\,\mu_{dh0}^{6} + 32\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4} - 72\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+48\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} + 48\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4} \\
-96\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} + 48\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} + 48\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-72\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 144\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-216\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 72\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 48\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-48\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 32\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} - 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+2\,\eta_d^{8}\mu_{cd0}^{7}\mu_{de0}^{4} - 6\,\eta_d^{8}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} + 6\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-2\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} + 144\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} - 144\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+24\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 16\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4} - 48\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+48\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} - 16\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}\,\mu_{dh0}^{6} + 96\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-72\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 8\,\eta_d^{2}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4} + 8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} \\
+6\,\eta_d^{8}\mu_{cd0}^{6}\mu_{de0}^{4} - 12\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 6\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+48\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} - 24\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 64\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-140\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 88\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 12\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} \\
+48\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} - 24\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 60\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-120\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 60\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} + 8\,\eta_d^{2}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4} \\
+6\,\eta_d^{8}\mu_{cd0}^{5}\mu_{de0}^{4} - 6\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 80\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-116\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 36\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 132\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-194\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 62\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} + 32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 2\,\eta_d^{8}\mu_{cd0}^{4}\mu_{de0}^{4} + 32\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-24\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 2\,\eta_d^{6}\mu_{cd0}^{6}\mu_{de0}^{4} - 6\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+6\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 2\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} + 72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-72\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 12\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{2}\mu_{de0}^{2}\mu_{dh0}^{4} + 8\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} \\
-24\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} + 24\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}\,\mu_{dh0}^{6} \\
+32\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} - 24\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} + 2\,\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4} \\
+2\,\eta_d^{6}\mu_{cd0}^{5}\mu_{de0}^{4} - 4\,\eta_d^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 2\,\eta_d^{6}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
+16\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} - 34\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} + 20\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-2\,\eta_d^{4}\eta_w^{2}\mu_{cd0}^{2}\mu_{de0}\,\mu_{dh0}^{6} + 12\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} - 24\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
+12\,\eta_d^{2}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4}
\end{array}. \tag{B.16}
$$

$$
P_9 = \begin{array}{c}
-12\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{8}\mu_{de0}^{4} + 24\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-48\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4} + 72\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-32\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{7}\mu_{de0}^{4} + 32\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} - 72\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+72\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 96\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+72\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{4}\eta_w^{8}\mu_{cd0}^{6}\mu_{de0}^{4} - 48\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+24\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{7}\mu_{de0}^{4} + 16\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} - 96\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} + 48\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-24\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{7}\mu_{de0}^{4} + 48\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-24\,\eta_d^{4}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4} - 12\,\eta_d^{8}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} - 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+32\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 32\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} \\
+8\,\eta_d^{6}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 96\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} + 136\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-40\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 12\,\eta_d^{4}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4} - 40\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+40\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} + 16\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-120\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} + 112\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 16\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-88\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} + 64\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 8\,\eta_d^{2}\eta_w^{8}\mu_{cd0}^{5}\mu_{de0}^{4} \\
-8\,\eta_d^{8}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{4} - 48\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{4} + 24\,\eta_d^{6}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} \\
-8\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{6}\mu_{de0}^{4} + 16\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 8\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} \\
-48\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{4} + 24\,\eta_d^{4}\eta_w^{6}\mu_{cd0}^{3}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{6}\mu_{de0}^{4} \\
+24\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{3}\mu_{dh0}^{2} - 12\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{2}\eta_w^{8}\mu_{cd0}^{4}\mu_{de0}^{4} \\
-8\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{5}\mu_{de0}^{4} + 8\,\eta_d^{6}\eta_w^{2}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 24\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+32\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2} - 8\,\eta_d^{4}\eta_w^{4}\mu_{cd0}^{3}\mu_{de0}^{2}\mu_{dh0}^{4} - 8\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{5}\mu_{de0}^{4} \\
+8\,\eta_d^{2}\eta_w^{6}\mu_{cd0}^{4}\mu_{de0}^{3}\mu_{dh0}^{2}
\end{array}. \tag{B.17}
$$

$$P_{10} = \begin{array}{c} 8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^8\mu_{de0}{}^4 - 8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^7\mu_{de0}{}^3\mu_{dh0}{}^2 + 32\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^7\mu_{de0}{}^4 \\ -24\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^3\mu_{dh0}{}^2 + 8\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^7\mu_{de0}{}^4 + 48\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 \\ -24\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 24\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 + 32\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 12\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^7\mu_{de0}{}^4 - 14\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^6\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +2\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^2\mu_{dh0}{}^4 + 24\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 + 16\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^7\mu_{de0}{}^4 \\ -16\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^3\mu_{dh0}{}^2 + 8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^4 + 36\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^6\mu_{de0}{}^4 \\ -28\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 2\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 + 8\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^4 \\ +64\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 - 44\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 10\,\eta_d{}^4\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 \\ +36\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^4 - 14\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 80\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -36\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 22\,\eta_d{}^4\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 + 12\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^4 \\ +2\,\eta_d{}^8\eta_w{}^2\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 - 2\,\eta_d{}^8\eta_w{}^2\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 + 32\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^4 \\ -8\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^3\mu_{de0}{}^3\mu_{dh0}{}^2 + 12\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^6\mu_{de0}{}^4 - 14\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 \\ +2\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 + 12\,\eta_d{}^4\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^4 + 8\,\eta_d{}^4\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 \\ -8\,\eta_d{}^4\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 2\,\eta_d{}^8\eta_w{}^2\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 12\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -4\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 - 2\,\eta_d{}^6\eta_w{}^4\mu_{cd0}{}^3\mu_{de0}{}^2\mu_{dh0}{}^4 + 16\,\eta_d{}^4\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 \\ -10\,\eta_d{}^4\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 + 2\,\eta_d{}^2\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 \end{array}. \tag{B.18}$$

$$P_{11} = \begin{array}{c} -2\,\eta_d{}^8\eta_w{}^8\mu_{cd0}{}^8\mu_{de0}{}^4 - 8\,\eta_d{}^8\eta_w{}^8\mu_{cd0}{}^7\mu_{de0}{}^4 - 12\,\eta_d{}^8\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 \\ -8\,\eta_d{}^8\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 - 8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^7\mu_{de0}{}^4 + 4\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^3\mu_{dh0}{}^2 \\ -4\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^7\mu_{de0}{}^4 - 2\,\eta_d{}^8\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^4 - 24\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 \\ +8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 - 16\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 - 24\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 \\ +4\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 - 20\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 - 8\,\eta_d{}^8\eta_w{}^6\mu_{cd0}{}^4\mu_{de0}{}^4 \\ -4\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 + 2\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^2\mu_{dh0}{}^4 - 8\,\eta_d{}^6\eta_w{}^8\mu_{cd0}{}^4\mu_{de0}{}^4 \\ -8\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^6\mu_{de0}{}^4 + 4\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^3\mu_{dh0}{}^2 - 2\,\eta_d{}^4\eta_w{}^8\mu_{cd0}{}^6\mu_{de0}{}^4 \\ -4\,\eta_d{}^8\eta_w{}^4\mu_{cd0}{}^4\mu_{de0}{}^3\mu_{dh0}{}^2 - 8\,\eta_d{}^6\eta_w{}^6\mu_{cd0}{}^5\mu_{de0}{}^4 - 4\,\eta_d{}^4\eta_w{}^8\mu_{cd0}{}^5\mu_{de0}{}^4 \end{array}. \tag{B.19}$$

Data availability

All data, models, and code generated or used during the study appear in the submitted article.
No data was used for the research described in the article.

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